Simultaneous Localization and Mapping (SLAM)

Lecture 02

Recall

Discrete Kalman Filter

Prediction

- Predicted state $\hat{x}(k)^{-} = F(k)\hat{x}(k-1) + Bu(k-1)$
- Predicted estimate covariance $P(k)^{-} = FP(k-1)F^{T} + Q$

Observation

- Innovation $\tilde{y}(k) = z(k) - H\hat{x}(k)^{-}$
- Innovation covariance $S(k) = HP(k)^{-}H^{T} + R$

Update

- Optimal Kalman gain $K(k) = P(k)^{-}HS(k)^{-1}$
- Updated state estimate $\hat{x}(k) = \hat{x}(k)^{-} + K(k)\tilde{y}(k)$
- Updated estimate covariance $P(k) = (I - K(k)H)P(k)^{-}$

Recall

Discrete Kalman Filter

Prediction

(1) Project the state ahead

$$\hat{x}(k)^{-} = F(k)\hat{x}(k-1) + Bu(k-1)$$

(2) Project the error covariance ahead

$$P(k)^{-} = FP(k-1)F^{T} + Q$$

Observation and Update

(1) Compute the Kalman gain

$$K(k) = P(k)^{-}H^{T}(HP(k)^{-}H^{T} + R)^{-1}$$

(2) Update estimate with measurement z(k)

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)[z(k) - H\hat{x}(k)^{-}]$$

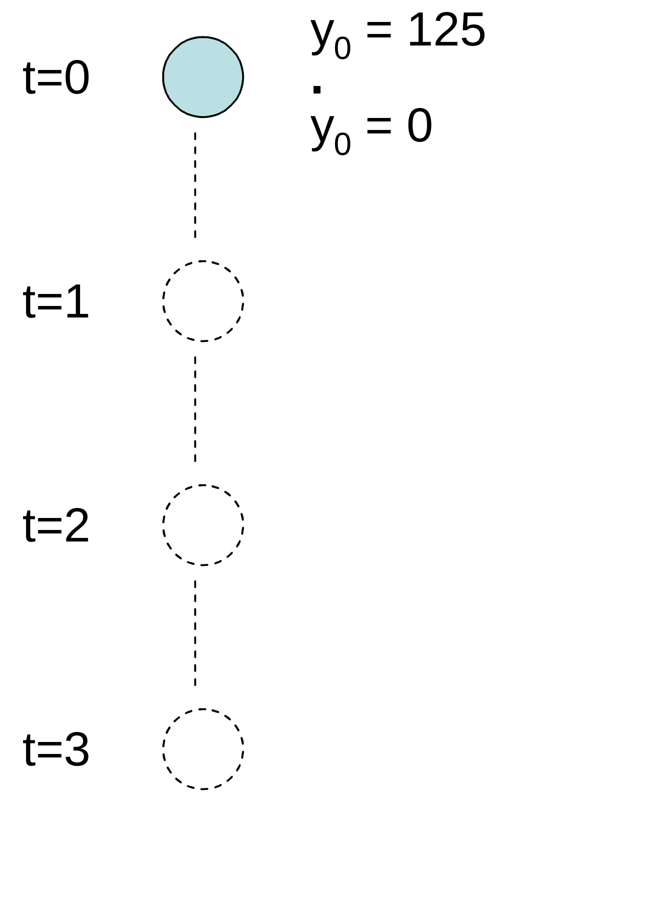
(3) Update error covariance

$$P(k) = (I - K(k)H)P(k)^{-}$$

Initial estimates for

$\hat{x}(k-1)$ & $P(k-1)$

Another Example

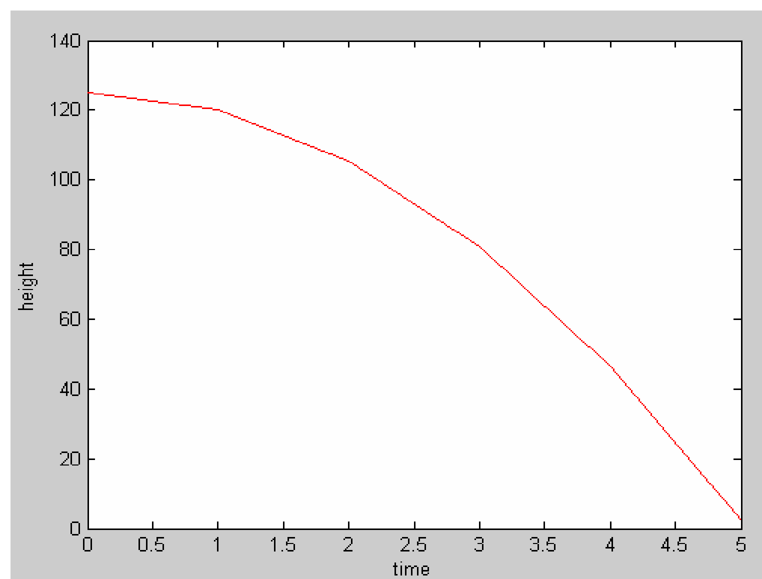


Kinematic Equations

$$y - y_0 = \dot{y}_0 \Delta t + \frac{1}{2} a (\Delta t)^2$$

$$\dot{y} = \dot{y}_0 + a \Delta t$$

Position (from model)



Process Model

Process Model

$$y(k+1) = y(k) + \dot{y}(k)\Delta t + \frac{1}{2}a(\Delta t)^2$$

$$\dot{y}(k+1) = \dot{y}(k) + a\Delta t$$

where $\begin{bmatrix} y(k+1) \\ \dot{y}(k+1) \end{bmatrix} = x(k+1)$ and $\begin{bmatrix} y(k) \\ \dot{y}(k) \end{bmatrix} = x(k)$

so

$$x(k+1) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} a$$

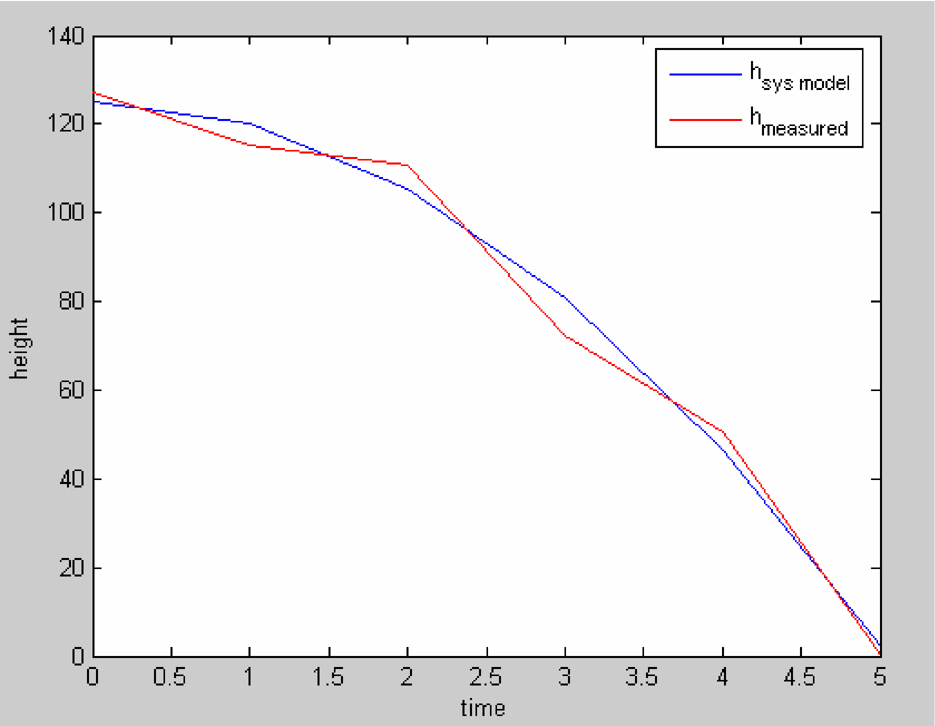
Observation Model

Observation Model

$$z(k) = Hx(k) + v(k)$$

where $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ because z is a measurement of the height directly

z =[127.0	115.3	110.9
72.4	50.7	0.3]



Kalman Filter

$$R = 1$$
$$\Delta t = 1$$

Initial Estimates

$$\hat{x}(k-1) = \begin{bmatrix} y(k-1) \\ \dot{y}(k-1) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \qquad P(k-1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Prediction

$$\hat{x}(k)^- = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \hat{x}(k-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * -9.81 \qquad P(k)^- = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P(k-1) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

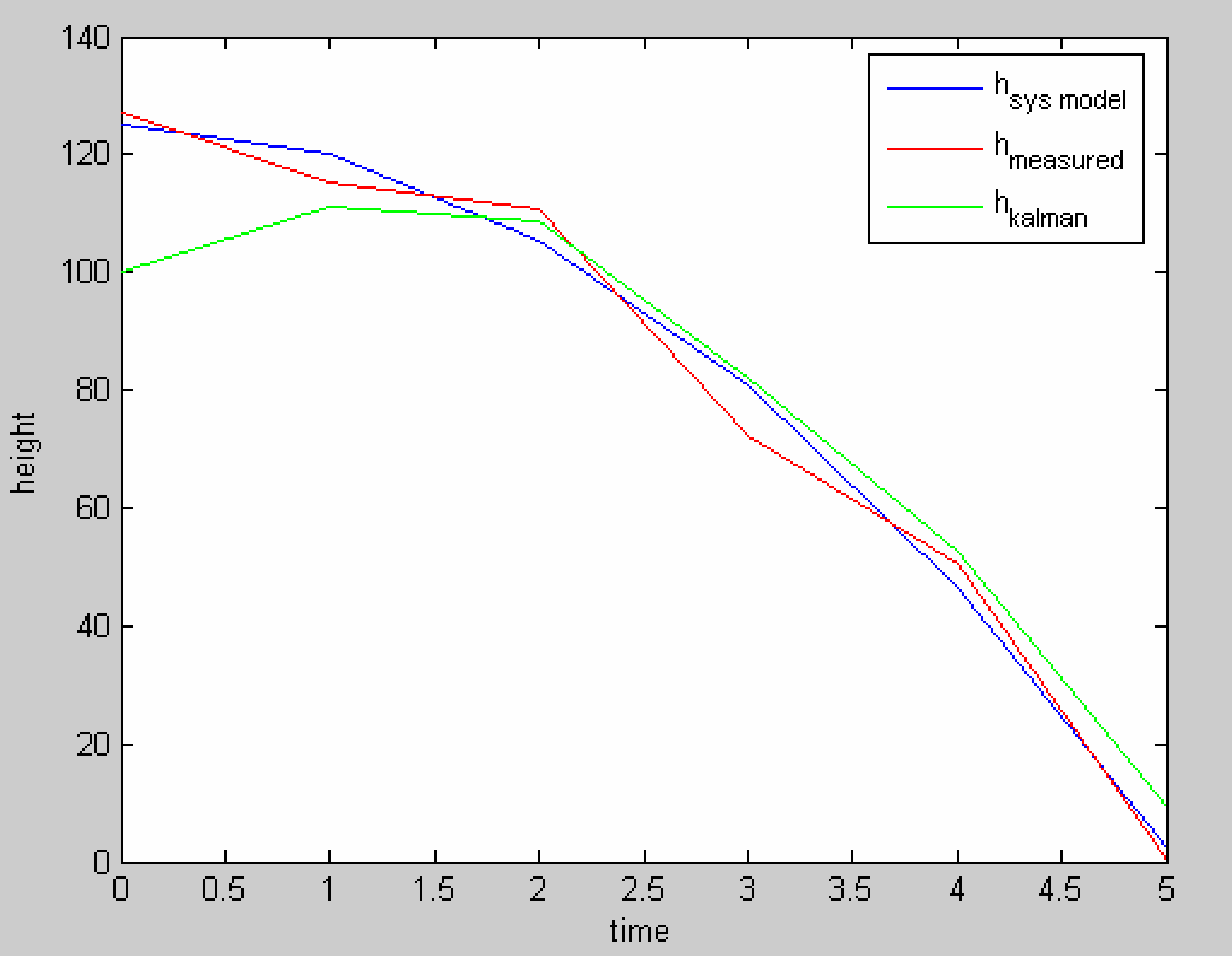
Observation and Update

$$K(k) = P(k)^- H^T (HP(k)^- H^T + R)^{-1}$$

$$\hat{x}(k) = \hat{x}(k)^- + K(k)[z(k) - H\hat{x}(k)^-]$$

$$P(k) = (I - K(k)H)P(k)^-$$

Kalman Filter



Non-linear Systems?

Non-linear Systems

Kalman Filter

- Limited to a linear assumption
- A non-linearity in a system can be associated with either the process model or the observation model (or both)

Extended Kalman Filter

- Process and observation models can both be non-linear

$$x(k) = f\left(x(k-1), u(k-1), w(k-1)\right)$$
$$z(k) = h\left(x(k), v(k)\right)$$

where f and h are non-linear functions

Extended Kalman Filter

Noise Parameters

- In practice, one does not know the noise values w(k) and v(k) at every time step
- Instead, the state and measurement vector are approximated without them

$$\tilde{x}(k) = f\left(\hat{x}(k-1), u(k), 0\right)$$
$$\tilde{z}(k) = h\left(\tilde{x}(k), 0\right)$$

where $\hat{x}(k)$ is some a posteriori estimate of the state

EKF

$$\tilde{x}(k) = f\left(\hat{x}(k-1), u(k), 0\right)$$
$$\tilde{z}(k) = h\left(\tilde{x}(k), 0\right)$$

To estimate a non-linear process, we need to linearize system at the current state

$$x(k) = \tilde{x}(k) + A\left(x(k-1) - \hat{x}(k-1)\right) + Ww(k-1)$$
$$z(k) = \tilde{z}(k) + J_h\left(x(k) - \tilde{x}(k)\right) + Vv(k)$$

$x(k), z(k)$	: actual state and measurement vectors
$\tilde{x}(k), \tilde{z}(k)$	: approximate state and measurement vectors
$\hat{x}(k)$	: a posteriori estimate of the state at step k
$w(k), v(k)$	: process and measurement noise
A	: Jacobian matrix of partial derivatives of f w.r.t. x
W	: Jacobian matrix of partial derivatives of f w.r.t. w
J_h	: Jacobian matrix of partial derivatives of h w.r.t. x
V	: Jacobian matrix of partial derivatives of h w.r.t. v

EKF

Let's define new notations for the prediction and measurement error

$$\tilde{e}_x(k) = x(k) - \tilde{x}(k) \qquad \tilde{e}_z(k) = z(k) - \tilde{z}(k)$$

Therefore, we have

$$\tilde{e}_x(k) \approx A\big(x(k-1) - \hat{x}(k-1)\big) + \varepsilon(k)$$

$$\tilde{e}_z(k) \approx J_h \tilde{e}_x(k) + \eta(k)$$

where $\varepsilon(k)$ and $\eta(k)$ represent new noise var.

$$p(\varepsilon(k)) \sim N\left(0, WQ(k)W^T\right)$$

$$p(\eta(k)) \sim N\left(0, VR(k)V^T\right)$$

The above equations are linear and closely resemble the difference equations from the discrete KF. Therefore, we could use a 2nd Kalman filter to estimate the prediction error

$$\hat{e}(k) = e(k)^- + K_k\big(z(k) - \tilde{z}(k)\big) = K_k \tilde{e}_z(k)$$ **(update equation)**

$$\hat{e}_x(k) = \boxed{\hat{x}(k)} - \tilde{x}(k)$$

This is what we are trying to find!!

EKF

Rearranging the predicted error estimate yields

$$\hat{e}_x(k) = \hat{x}(k) - \tilde{x}(k) \quad \Longrightarrow \quad \hat{x}(k) = \tilde{x}(k) + \hat{e}_x(k)$$

Plugging in from the previous slide

$$\hat{x}(k) = \tilde{x}(k) + K_k \tilde{e}_z(k) \quad \Longrightarrow \quad \boxed{\hat{x}(k) = \tilde{x}(k) + K_k\left(z(k) - \tilde{z}(k)\right)}$$

The equation above can now be used in the measurement update in EKF!

EKF

Prediction

- Predicted state $\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$
- Predicted estimate covariance $P(k)^{-} = F(k)P(k-1)F(k)^{T} + W(k)Q(k-1)W(k)^{T}$

Observation

- Innovation $\tilde{y}(k) = z(k) - H$ where H is the sensor model
- Innovation covariance $S(k) = J_{h}(k)P(k)^{-}J_{h}(k)^{T} + V(k)R(k)V(k)^{T}$

Update

- Optimal Kalman gain $K(k) = P(k)^{-}J_{h}(k)^{T}S(k)^{-1}$
- Updated state estimate $\hat{x}(k) = \hat{x}(k)^{-} + K(k)\tilde{y}(k)$
- Updated estimate covariance $P(k) = \left(I - K(k)J_{h}(k)\right)P(k)^{-}$

EKF

Prediction

(1) Project the state ahead

$$\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

(2) Project the error covariance ahead

$$P(k)^{-} = F(k)P(k-1)F(k)^{T} + W(k)Q(k-1)W(k)^{T}$$

Observation and Update

(1) Compute the Kalman gain

$$K(k) = P(k)^{-} J_{h}(k)^{T}\left(J_{h}(k)P(k)^{-} J_{h}(k)^{T} + V(k)R(k)V(k)^{T}\right)$$

(2) Update estimate with measurement z(k)

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\left[z(k) - H\right]$$

(3) Update error covariance

$$P(k) = \left(I - K(k)J_{h}(k)\right)P(k)^{-}$$

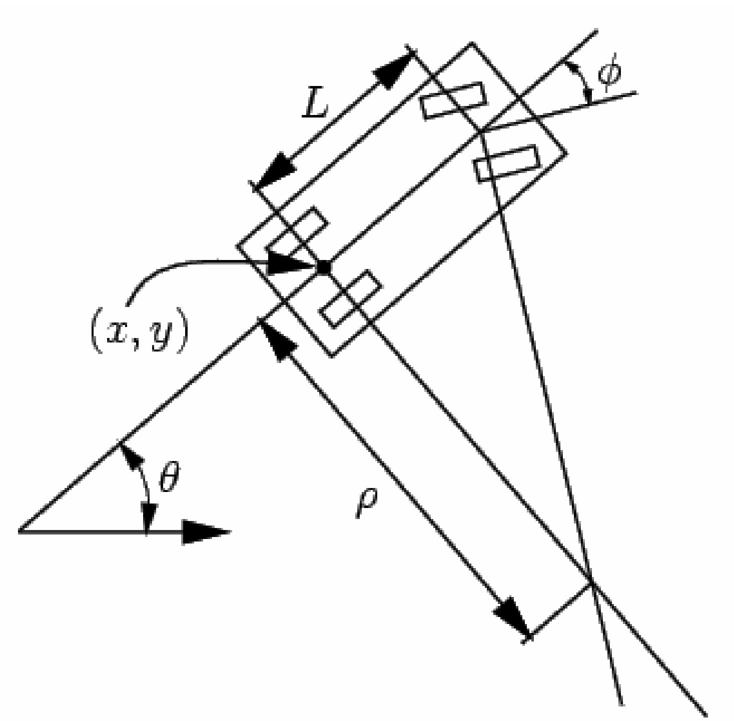
Initial estimates for

$\hat{x}(k-1)$ & $P(k-1)$

A EKF in Action

An Example…

Simple Robot Model



Kinematic Equations

$$\dot{x} = V\cos\theta$$

$$\dot{y} = V\sin\theta$$

$$\dot{\theta} = \frac{V\tan\phi}{L}$$

Non-linear!

Simple Robot Model

Kinematic Equations

$$\dot{x} = V\cos\theta$$

$$\dot{y} = V\sin\theta$$

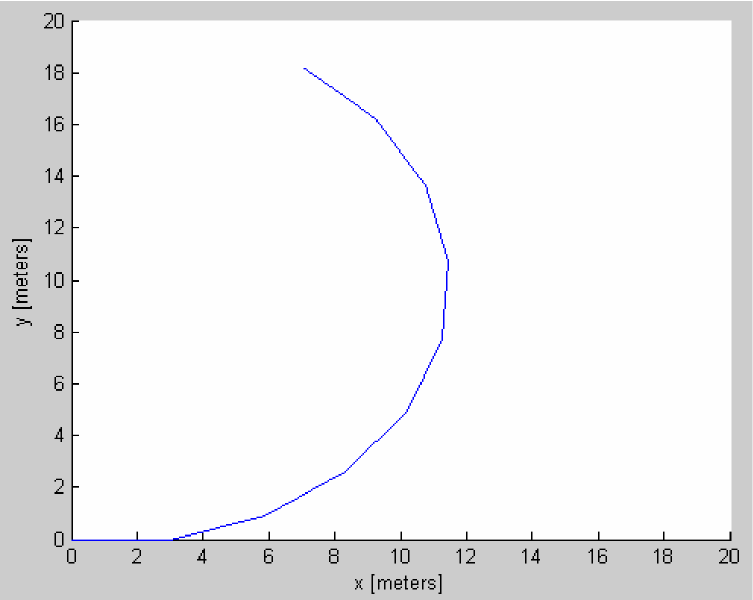
$$\dot{\theta} = \frac{V\tan\phi}{L}$$

f(x,u,w)

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \theta(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\theta(k) \\ y(k) + \Delta t V(k)\sin\theta(k) \\ \dfrac{\Delta t V(k)\tan\phi(k)}{L} \end{bmatrix}$$

Assumptions

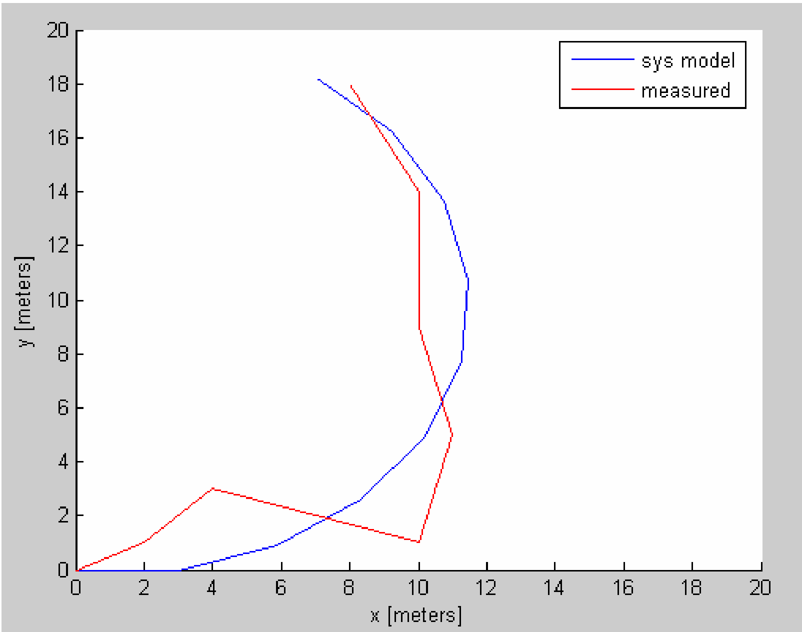
- System inputs
  - Velocity (assumed constant, vel=3)
  - Steering angle (ϕ)
- Δt is fixed and equal to 1
- L=1
- 10 iterations (N=10)

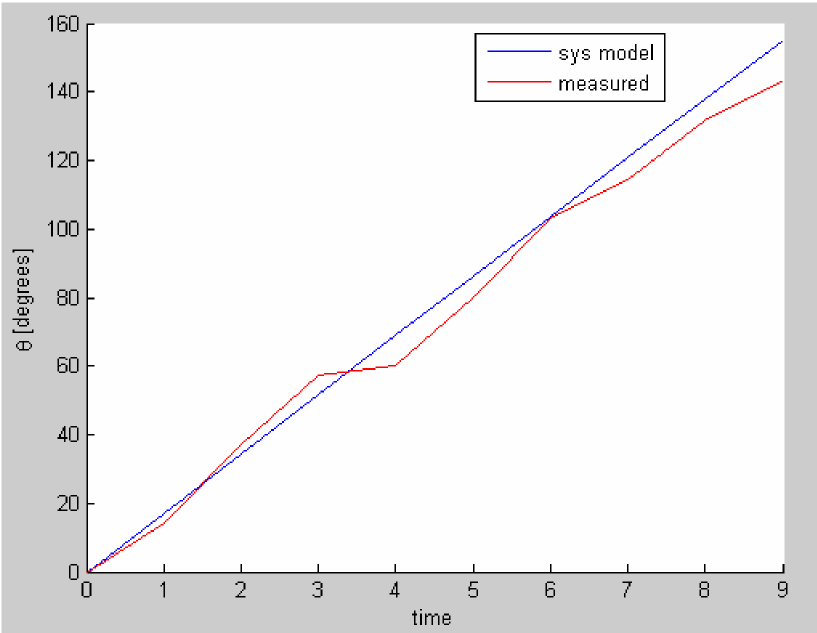


Observation Model

Measurements are taken from an overhead camera, and thus x, y, and θ can be measured directly

$$z(k) = h\big(x(k), v(k)\big) \quad \Rightarrow \quad z(k) = \begin{bmatrix} x(k) + v_x \\ y(k) + v_y \\ \theta(k) + v_\theta \end{bmatrix}$$





EKF Example

Prediction

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$ from robot model

$$P(k)^- = \underline{F(k)}P(k-1)\underline{F(k)}^T + \underline{W(k)}Q(k-1)\underline{W(k)}^T$$

$$x(k+1) = f\left(x(k), u(k), w(k)\right) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\theta(k) \\ y(k) + \Delta t V(k)\sin\theta(k) \\ \theta(k) + \dfrac{\Delta t V(k)\tan\phi(k)}{L} \end{bmatrix}$$

Need to calculate Jacobians!

$$F(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial \theta} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial \theta} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -V\sin\theta \\ 0 & 1 & V\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$W(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w_x} & \dfrac{\partial f_1}{\partial w_y} & \dfrac{\partial f_1}{\partial w_\theta} \\ \dfrac{\partial f_2}{\partial w_x} & \dfrac{\partial f_2}{\partial w_y} & \dfrac{\partial f_2}{\partial w_\theta} \\ \dfrac{\partial f_3}{\partial w_x} & \dfrac{\partial f_3}{\partial w_y} & \dfrac{\partial f_3}{\partial w_\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EKF Example

Kalman Gain

$$K(k) = P(k)^- J_h(k)^T \left(J_h(k)P(k)^- J_h(k)^T + V(k)R(k)V(k)^T\right)^{-1}$$

$$z(k) = h\left(x(k), v(k)\right) = \begin{bmatrix} x(k)+v_x \\ y(k)+v_y \\ \theta(k)+v_\theta \end{bmatrix}$$

Need to calculate Jacobians!

$$J_h(k) = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial \theta} \\ \frac{\partial h_3}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V(k) = \begin{bmatrix} \frac{\partial h_1}{\partial v_x} & \frac{\partial h_1}{\partial v_y} & \frac{\partial h_1}{\partial v_\theta} \\ \frac{\partial h_2}{\partial v_x} & \frac{\partial h_2}{\partial v_y} & \frac{\partial h_2}{\partial v_\theta} \\ \frac{\partial h_3}{\partial v_x} & \frac{\partial h_3}{\partial v_y} & \frac{\partial h_3}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EKF Example

Measurement Update

$$\hat{x}(k) = \hat{x}(k)^- + K(k)\left(z(k) - H\right)$$

$$P(k) = \left(I - K(k)J_h(k)\right)P(k)^-$$