Jacobians

Definition: map differential changes from one space to another

Recall for 2-link: $$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} + a_1 c_1 & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} + a_1 s_1 & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Consider general tool transformation matrix for an n-link manipulator with joint variables $q_1 \cdots q_n$

$$T_n^0(\boldsymbol{q}) = \begin{bmatrix} R_n^0(\boldsymbol{q}) & o_n^0(\boldsymbol{q}) \\ \boldsymbol{0} & 1 \end{bmatrix}$$

where
$\boldsymbol{q} = (q_1, \cdots q_n)^T$: joint variable vector
o_n^0: end-effector position
R_n^0: end-effector orientation

Objective: Relate linear and angular velocities of end-effector to vector of joint velocities

Define the following:

$$S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$$ End-effector's angular velocity vector ω_n^0

$$v_n^0 = \dot{o}_n^0$$ End-effector's linear velocity vector

Want: $$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J_n^0 \dot{\boldsymbol{q}}$$ where $$\boxed{J_n^0 = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}}$$ is called the **Manipulator Jacobian**

Angular Velocity: Can be shown that generally

$$J_\omega = [\rho_1 z_0 \quad \cdots \quad \rho_n z_{n-1}]$$

Where have
$\rho_i = 1$ if joint is revolute
$\rho_i = 0$ if joint is prismatic

Linear Velocity: Just take derivatives

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

Column i of J_v would be given by

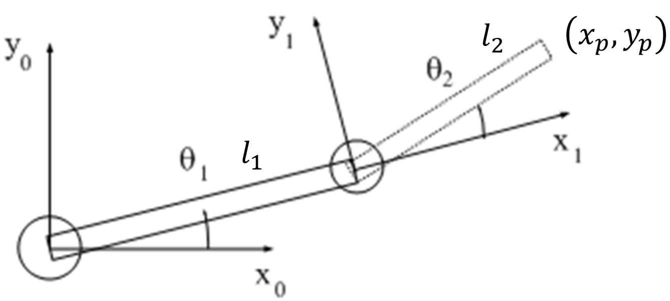
$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i} \qquad \text{where} \qquad J_v = [J_{v_1} \quad \cdots \quad J_{v_n}]$$

Sanity Check: Calculate Jacobian for 2-link planar manipulator

$$J_2^0 = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$ Hence, we have $$J_\omega = [\rho_1 z_0 \quad \cdots \quad \rho_2 z_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

We also have $$J_v = [J_{v_1} \quad \cdots \quad J_{v_n}] = \begin{bmatrix} \frac{\partial o_2^0}{\partial q_1} & \frac{\partial o_2^0}{\partial q_2} \end{bmatrix}$$

Recall our Tool Transformation Matrix:



$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} + a_1 c_1 & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} + a_1 s_1 & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Hence: $$o_2^0 = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix}$$

Thus $$\frac{\partial o_2^0}{\partial q_1} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ 0 \end{bmatrix}$$ and $$\frac{\partial o_2^0}{\partial q_2} = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \\ 0 \end{bmatrix}$$

$$J_2^0 = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \left[\begin{array}{cc} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{array}\right]$$

Since: $\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J_n^0 \dot{\boldsymbol{q}}$ then

$$v_2^0 = (-l_1 s_1 - l_2 s_{12})\dot{\theta}_1 - l_2 s_{12}\dot{\theta}_2$$

$$\omega_2^0 = (l_1 c_1 + l_2 c_{12})\dot{\theta}_1 + l_2 c_{12}\dot{\theta}_2$$