Robotics

Inverse Kinematics

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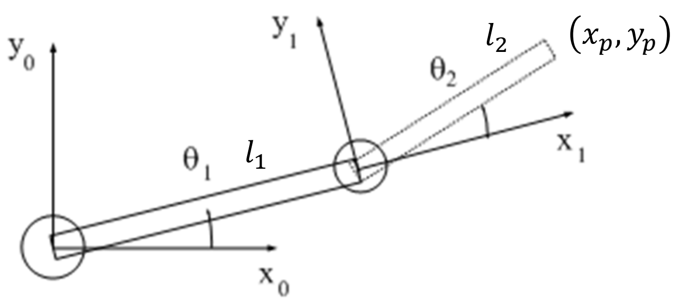
Big Question: Inverse kinematics (IK) asks what joint parameters are needed to place robot's end-effector at desired point?

Recall from study of 2-link planar manipulator:



$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Tool transformation matrix T_2^0 maps tool (end-effector) frame from $x_2 y_2$ to base frame $x_0 y_0$

Problem: Solve θ_1 and θ_2 as a function of x_p and y_p

Solution (Algebraic Approach):

Step 1: Solve for θ_2

From last column of Tool transformation matrix, and figure, see that

$$x_p^2 + y_p^2 = (l_1 c_1 + l_2 c_{12})^2 + (l_1 s_1 + l_2 s_{12})^2 = l_1^2 c_1^2 + l_2^2 c_{12}^2 + 2 l_1 l_2 c_1 c_{12} + l_1^2 s_1^2 + l_2^2 s_{12}^2 + 2 l_1 l_2 s_1 s_{12}$$

$$x_p^2 + y_p^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_1 c_{12} + 2 l_1 l_2 s_1 s_{12} = l_1^2 + l_2^2 + 2 l_1 l_2 [c_1 (c_1 c_2 - s_1 s_2) + s_1 (s_1 c_2 + s_2 c_1)]$$

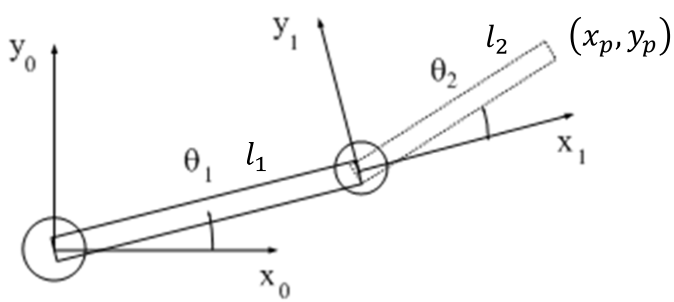
$$x_p^2 + y_p^2 = l_1^2 + l_2^2 + 2 l_1 l_2 [c_1^2 c_2 - c_1 s_1 s_2 + s_1^2 c_2 + s_1 s_2 c_1]$$

$$x_p^2 + y_p^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2 \qquad (1)$$

Consequently from (1) have

$$\theta_2 = \cos^{-1} \frac{x_p^2 + y_p^2 - l_1^2 - l_2^2}{2 l_1 l_2} \tag{2}$$

Inverse cos is inaccurate with small angles. Hence, re-express (2) using inverse tan:



$$sin^2\theta_2 + cos^2\theta_1 = 1$$

$$\sin\theta_2 = \pm\{1 - cos^2\theta_2\}^{1/2} \tag{3}$$

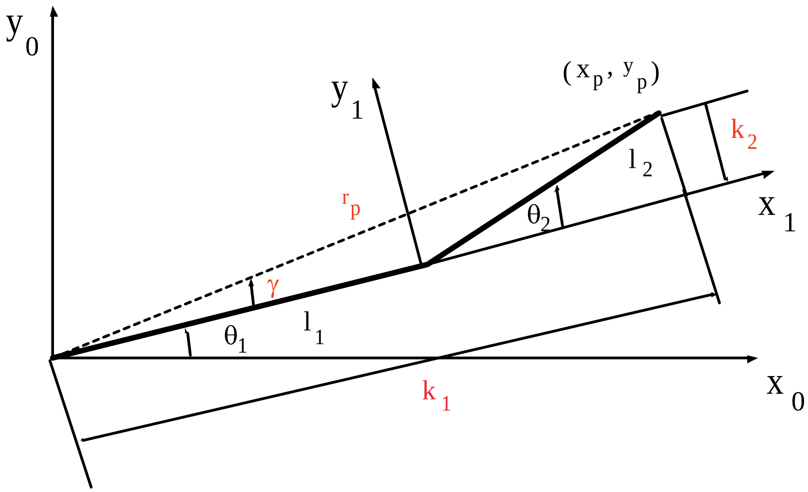
Observe that (3) has two solutions representing elbow up/down configuration

$$\theta_2 = atan2\left(\pm\left\{1 - \left(\frac{x_p^2 + y_p^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)^2\right\}^{1/2}, \quad \frac{x_p^2 + y_p^2 - l_1^2 - l_2^2}{2 l_1 l_2} \right) \tag{4}$$

Sine part from (3)

Cosine part from (2)

Step 2: Solve for θ_1



Adding the variables in red, one has

$$k_1 = l_1 + l_2 \cos\theta_2$$

$$k_2 = l_2 \sin\theta_2$$

$$r_p^2 = k_1^2 + k_2^2$$

$$\gamma = \tan^{-1}\left(\frac{k_2}{k_1}\right)$$

$$k_1 = r_p \cos\gamma$$

$$k_2 = r_p \sin\gamma$$

One can derive the following:

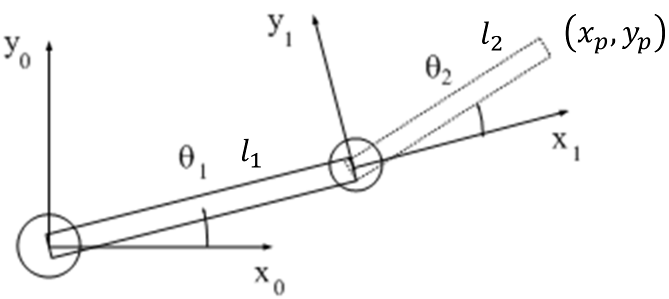
$$x_p = k_1 \cos\theta_1 - k_2 \sin\theta_1 \quad (5)$$

$$y_p = k_1 \sin\theta_1 + k_2 \cos\theta_1 \quad (6)$$

Substituting for k_1 and k_2 one can rewrite (5) and (6) to yield:

$$\theta_1 = atan2(y_p, x_p) - atan2(k_2, k_1) \quad (7)$$

Sanity Check:

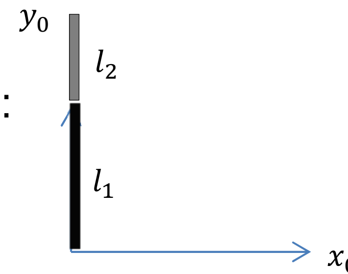


$$\theta_2 = atan2\left(\pm\left\{1-\left(\frac{x_p^2+y_p^2-l_1^2-l_2^2}{2l_1l_2}\right)^2\right\}^{1/2}, \frac{x_p^2+y_p^2-l_1^2-l_2^2}{2l_1l_2}\right)$$

Recall (4)

$$\theta_1 = atan2(y_p, x_p) - atan2(k_2, k_1)$$

Recall (7)



Case 1: If $(x_p, y_p) \triangleq (l_1 + l_2, 0)$ then? Can envision should have $\theta_1 = \theta_2 = 0$:

With $x_p = l_1 + l_2$ and $y_p = 0$ then (4) yields:

$$\theta_2 = atan2\left(\pm\left\{1-\left(\frac{(l_1+l_2)^2+0-l_1^2-l_2^2}{2l_1l_2}\right)^2\right\}^{1/2}, \frac{(l_1+l_2)^2+0-l_1^2-l_2^2}{2l_1l_2}\right)$$

$$\theta_2 = atan2\left(\pm\left\{1-\left(\frac{(2l_1l_2)}{2l_1l_2}\right)^2\right\}^{1/2}, \frac{(2l_1l_2)}{2l_1l_2}\right) = atan2(0,1)$$

or, $\theta_2 = 0$

Also, have from (5):

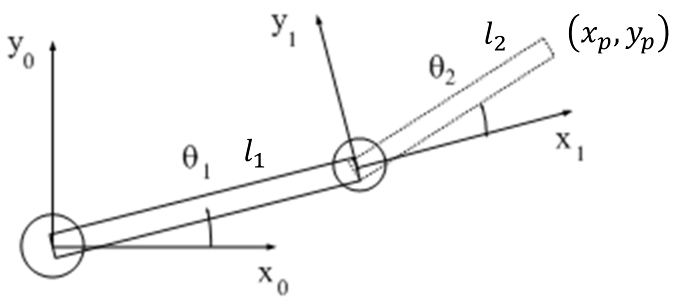
QED

$$\theta_1 = atan2(0, l_1 + l_2) - atan2(0, l_1 + l_2)$$

Hence, $\theta_1 = 0$

P.Oh added 04/08/19

Another Sanity Check:

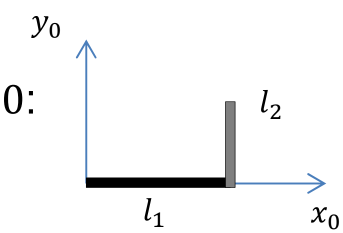


$$\theta_2 = atan2\left(\pm\left\{1-\left(\frac{x_p^2+y_p^2-l_1^2-l_2^2}{2l_1l_2}\right)^2\right\}^{1/2}, \frac{x_p^2+y_p^2-l_1^2-l_2^2}{2l_1l_2}\right)$$

$$\theta_1 = atan2(y_p, x_p) - atan2(k_2, k_1)$$

$$k_1 = l_1 + l_2\cos\theta_2$$

$$k_2 = l_2\sin\theta_2$$



Case 2: If $(x_p, y_p) \triangleq (l_1, -l_2)$ then? Can envision should have $\theta_1 = 0, \theta_2 = 90$:

With $x_p = l_1$ and $y_p = -l_2$ then (4) yields:

$$\theta_2 = atan2\left(\pm\left\{1-\left(\frac{l_1{}^2+l_2{}^2-l_1^2-l_2^2}{2l_1l_2}\right)^2\right\}^{1/2}, \frac{l_1{}^2+l_2{}^2+0-l_1^2-l_2^2}{2l_1l_2}\right)$$

$$\theta_2 = atan2\left(\pm\{1-(0)^2\}^{1/2}, \frac{(0)}{2l_1l_2}\right) = atan2(\pm 1, 0)$$

or, $\theta_2 = \pm 90$

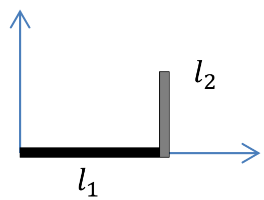
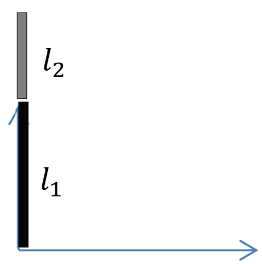
Choose -90
i.e. negative root

Also, have from (5):

$$\theta_1 = atan2(-l_2, l_1) - atan2(l_2, l_1)$$

Hence, $\theta_1 = 0$

Other configurations can be shown, yielding required θ_1 and θ_2

Sketch	x_p	y_p	θ_1	θ_2
	l_1	l_2	0	90 deg
	0	$l_1 + l_2$	90 deg	0