Robotics

Forward Kinematics, Denavit-Hartenberg, Transformation Matrices

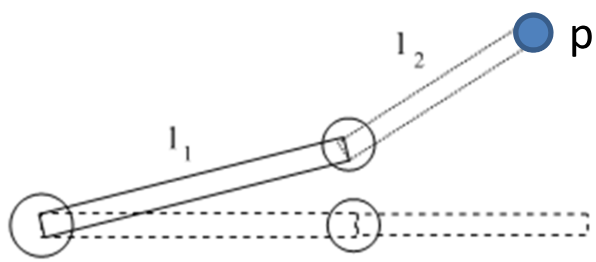
Reference: lectureDenavitHartenberg-103116.pptx

Forward Kinematics

Big Question: Want robot's end-effector at desired point. What joint angles are needed?

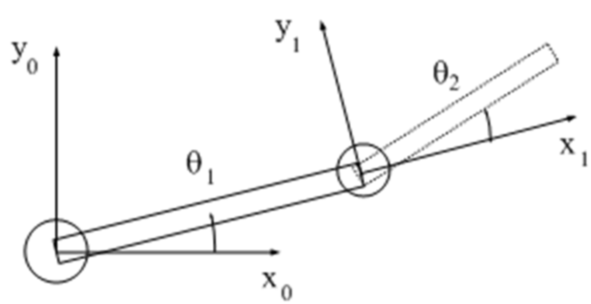
- Inverse Kinematics is the topic that answers this
- Solution (if any) likely is non-linear, transcendental, and often non-unique
- Demands inverting a (transformation) matrix – multiple techniques
- Solution begins by understanding the transformation matrix
- So must first understand Forward Kinematics

Sanity Check: Given the 2-link planar manipulator below, what is the (x, y) position of the end-point p?

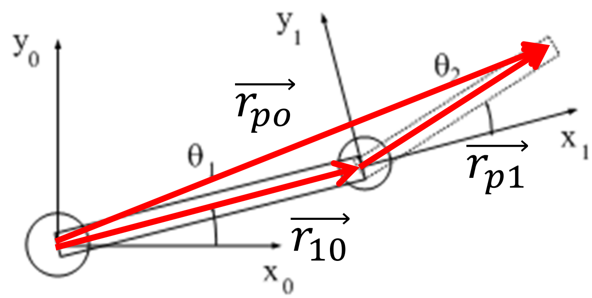


Solution: Recall Relative Reference Frames from Physics, Mechanics and Kinematics

Step 1: Assign a reference frame



Step 2: Apply vectors



$$\overrightarrow{r_{po}} = \overrightarrow{r_{10}} + \overrightarrow{r_{p1}}$$

Have the following:

$$\overrightarrow{r_{10}} = l_1 \cos\theta_1 \hat{\imath} + l_1 \sin\theta_1 \hat{\jmath}$$

$$\overrightarrow{r_{p1}} = l_2 \cos\theta_2 \widehat{i'} + l_2 \sin\theta_2 \widehat{j'}$$

Can show:

$$\boxed{\begin{aligned} x_{po} &= l_1 c_1 + l_2 c_{12} \\ y_{po} &= l_1 s_1 + l_2 s_{12} \end{aligned}}$$

where

$$c_k = \cos\theta_k \text{ and } s_k = \sin\theta_k$$

$$c_{12} = \cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \quad (1)$$

$$s_{12} = \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1$$

Take-away:

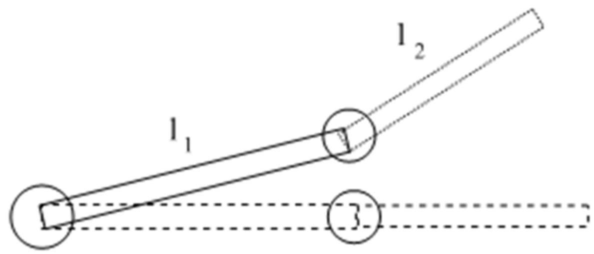
- The reference frames' poses are not unique – can position and orient anywhere
- Solving for absolute position for N-links, where N is large, is tedious

Denavit-Hartenberg Notation

Motivation:

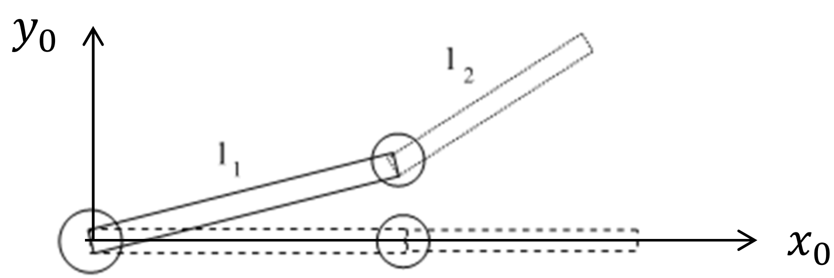
- Provide a standard notation for labeling and locating a robot's reference frames
- N links will yield N reference frames
- Standard is iterative and hence lends itself to computational implementation
- Caveats: Discrepant use of DH method – so read notation carefully
- Ultimately, DH is simply reference frames – but a lot of (robot) math is built on it

Illustrative Example: Given the 2-link planar manipulator below, apply DH notation and solve for end-effector's absolute location

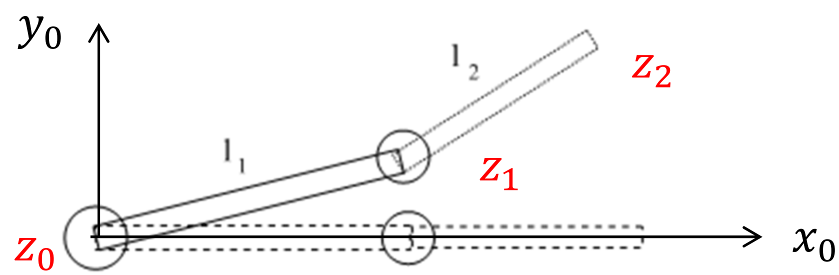


Step 1: Count n, number of links. Place (arbitrary) origin of base frame $x_0 y_0 z_0$

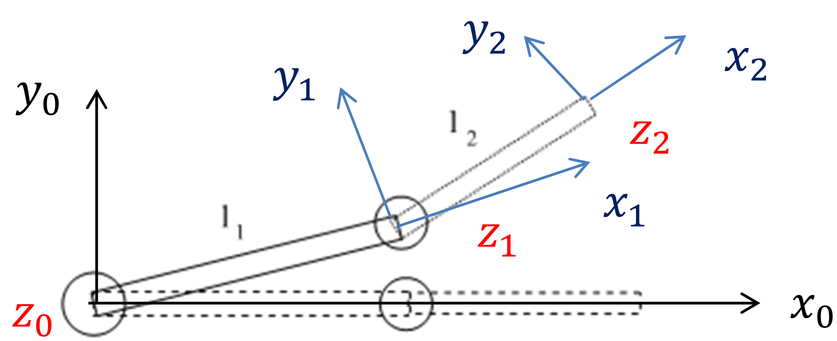
$n = 2$ links



Step 2: Label and locate joint axes $z_0, z_1 \cdots z_{n-1}$ and end-effector frame z_n



Step 3: Position axes $x_1, x_2 \cdots x_n$ along link (observing right-hand reference frame)



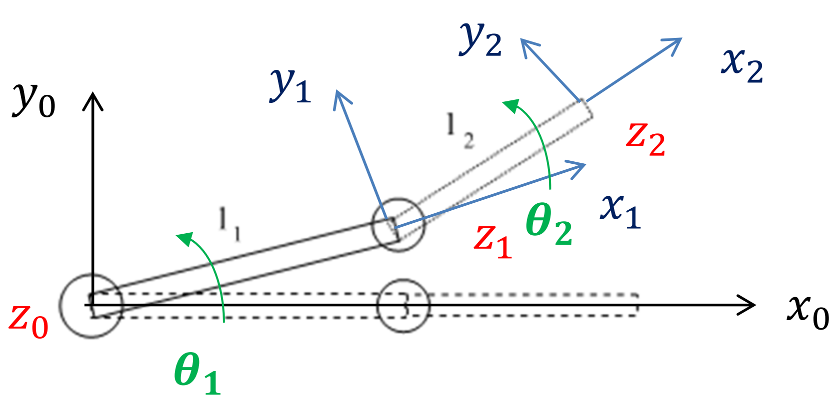
Step 4: Identify link parameters to DH definitions

$a_i \triangleq$ distance along x_i from O_i to the intersection of x_i and z_{i-1} axes

$d_i \triangleq$ distance along z_{i-1} from O_{i-1} to the intersection of x_i and z_{i-1} axes

$\alpha_i \triangleq$ angle between z_{i-1} and z_{i-1} measured about x_i (right-hand rule)

$\theta_i \triangleq$ angle between x_{i-1} and x_{i-1} measured about z_{i-1} (right-hand rule)



Link k	θ_k	d_k	α_k	a_k
1	θ_1	0	0	l_1

Link k	θ_k	d_k	α_k	a_k
2	θ_2	0	0	l_2

This table is the DH notation for the 2-link planar manipulator

NB: If joint k is revolute (like above), then $d_k = 0$. If joint k is prismatic, then $a_k = 0$

This video may be helpful for visualization https://www.youtube.com/watch?v=rA9tm0gTln8

Homogeneous Transformation Matrix

- Assigning reference frames is non-unique
- But DH notation serves as a kind of "standard" by roboticists
- Much (robot) math (and simulators) builds upon using this DH "standard"
- Homogeneous Transformation Matrix and Tool Frame are examples

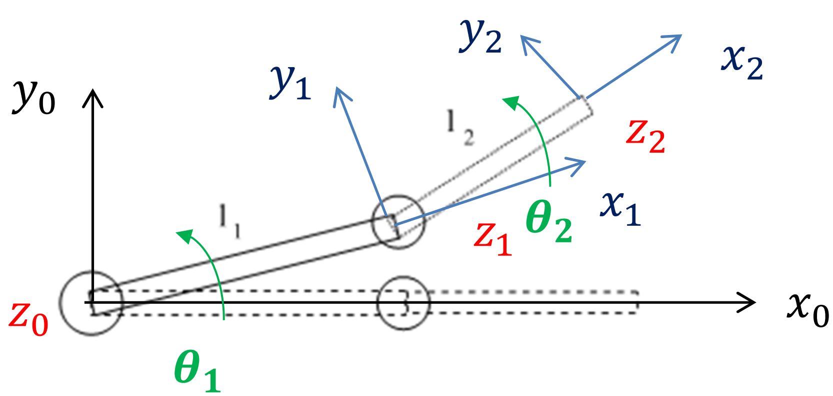
$$A_i \triangleq \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where A_i is called the Homogeneous Transformation Matrix and maps frame from $i-1$ to i

Tool Frame $\boldsymbol{T_n^0}$: More interested in knowing the transformation from robot's base (origin) to robot's end-effector (tool) frames

$$\boldsymbol{T_n^0 = A_1 A_2 \cdots A_n}$$

Sanity Check: Try homogeneous and tool transformation matrices with 2-link planar manipulator



Link k	θ_k	d_k	α_k	a_k
1	θ_1	0	0	l_1

Link k	θ_k	d_k	α_k	a_k
2	θ_2	0	0	l_2

Step 1: Solve individual homogeneous transformation matrices A_i

$$A_1 \triangleq \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & l_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & l_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A_2 \triangleq \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & l_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Footnote: hand-checked A1 and A2 10/31/16, 11/04/16; and 04/04/19.
Also see Spong pg. 84-85 Robot Modeling and Control 2006 Edition

Step 2: Solve product of homogeneous transformation matrices A_i

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $c_k = \cos\theta_k$ and $s_k = \sin\theta_k$

$$c_{12} = \cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$s_{12} = \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1$$

Note: The last column gives (x, y) position of end-point. Compare to (1) from before

$$\boxed{\begin{aligned} x_{po} &= l_1 c_1 + l_2 c_{12} \\ y_{po} &= l_1 s_1 + l_2 s_{12} \end{aligned}}$$

Takeaway: DH notation gives step-by-step analytical method to ultimately determine the (x, y) position of a tool (end-effector) with respect to base frame (i.e. forward kinematics)

Inverting the tool matrix, should enable one to determine the joint values needed to put tool at desired (x, y) and is called inverse kinematics which is topic of next lecture.

Footnote: hand-checked $T_2{}^0$ 10/31/16, 11/04/16; and 04/04/19.
Also see Spong pg. 84-85 Robot Modeling and Control 2006 Edition