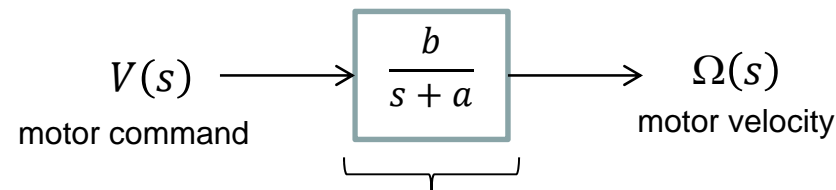


# Open-Loop Step Response

Recall that the dynamic equations of motion for a DC motor (for low inductances) can be modeled as a 1<sup>st</sup> order system.

Modeling can be also be represented by Block Diagrams and Laplace Transforms



$G_{OL}(s)$  : called Open Loop Transfer Function

Mathematically, this becomes:

$$G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} \quad (1)$$

or

$$(s+a)\Omega(s) = bV(s)$$

**Method 1:** Solve for  $\omega(t)$  with Ordinary Differential Equations:

(1) becomes:  $\dot{\omega}(t) + a\omega(t) = bv(t)$  (2)

Suppose one gives the motor a step input i.e. applies a velocity command

In time domain, step input is:

$$v(t) = \begin{cases} 0: t \leq 0 \\ M: t > 0 \end{cases} \quad (3)$$

Subbing (3) into (2) yields

$$\dot{\omega}(t) + a\omega(t) = bM \quad (4)$$

This first-order differential equation is solved using an integrating factor

$$\mu(x) = \exp \int^x p(t)dt$$

when one has equations of the form:

$$\dot{\omega} + p(t)\omega = g(t) \quad (5)$$

Thus comparing (4) and (5), say:

$$p(t) \triangleq a$$

and using integrating factor, one creates the equation:

$$e^{at}\{\dot{\omega} + a\omega\} = e^{at}bM$$

or

$$\frac{d}{dt}\{e^{at}\omega\} = bMe^{at} \quad (6)$$

Integrating (6) yields

$$e^{at}\omega = Mb \int e^{at} dt = Mb \left\{ \frac{1}{a} e^{at} + C \right\} \text{ where } C \text{ is unknown constant} \quad (7)$$

Multiplying (7) with the exponent yields:

$$\omega(t) = e^{-at} \left\{ \frac{M}{a} b e^{at} + D \right\}$$

or

$$\omega(t) = \frac{M}{a} b + D e^{-at} \text{ where } D \text{ is unknown constant} \quad (8)$$

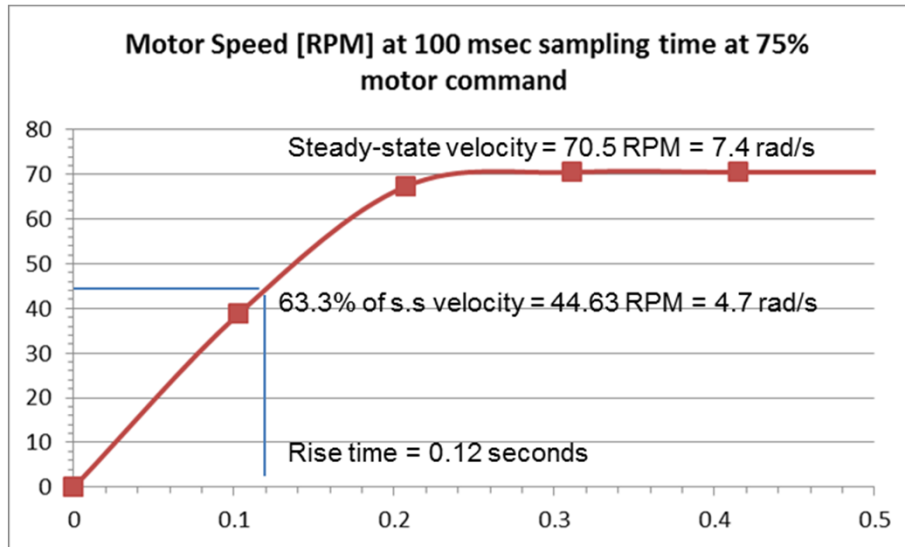
With initial conditions (IC), say that  $\omega(0) = 0$  then (8) becomes

$$0 = \frac{Mb}{a} + D \quad \text{or} \quad D = -\frac{Mb}{a} \quad (9)$$

Subbing (9) into (8), one gets general solution form

$$\omega(t) = \frac{Mb}{a} - \frac{Mb}{a} e^{-at} = \frac{Mb}{a} (1 - e^{-at}) \quad (10)$$

From M=75 step response lab of NXT DC motor, we saw a graph like:



$$\omega(t) = \omega_{ss}(1 - e^{-at}) \quad (11)$$

If we say that  $a \triangleq \frac{1}{\tau}$  then one has

$$\omega(t) = \omega_{ss} \left(1 - e^{-\frac{t}{\tau}}\right)$$

From graph, see  $\omega_{ss} = 70.5$  RPM

Brackets	$\omega$	Error [V]
$1 - e^{-1} = 0.63$	44.4	$70.5 - 44.4 = 26.1$ or 37% (i.e. <b>63%</b> of steady-state)
$1 - e^{-2} = 0.86$	60.6	$70.5 - 60.6 = 9.9$ or 14.0% (i.e. 86% of steady-state)
$1 - e^{-3} = 0.95$	67.0	$70.5 - 67.0 = 3.5$ or 5% (i.e. <b>95%</b> of steady-state)
$1 - e^{-4} = 0.86$	69.1	$70.5 - 69.1 = 1.4$ or 2% (i.e. 98% of steady-state)
$1 - e^{-5} = 0.63$	69.8	$70.5 - 69.8 = 0.7$ or 1% (i.e. <b>99%</b> of steady-state)

Comparing (10) and (11) we see that:

$$b = \frac{\omega_{ss}}{M\tau} \quad (12)$$

Subbing (11) and (12) into (1) yields:

$$G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} = \frac{\omega_{ss}}{M\tau} \frac{1}{s + \frac{1}{\tau}} \quad (13)$$

For  $\tau = 0.12$  and  $\omega_{ss} = 70.5$  RPM, and step input  $M = 75$  (13) becomes:

$$G_{OL}(s) = \frac{70.5}{75(0.12)} \frac{1}{s + \frac{1}{0.12}} = \frac{7.83}{s + 8.33} \quad (14)$$

**Method 2:** Solve for  $\omega(t)$  with Laplace Transforms:

Like before, and referring to (1), have

$$V(s) \longrightarrow \boxed{\frac{b}{s+a}} \longrightarrow \Omega(s) \qquad G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} \qquad \text{from (1)}$$

Like before, one has a step input i.e. velocity command, referring to (3)

$$v(t) = \begin{cases} 0: t \leq 0 \\ M: t > 0 \end{cases} \qquad \text{from (3)}$$

The Laplace Transform for a step input signal is given by:

$$V(s) = \frac{M}{s} \qquad (15)$$

Substituting (15) into (1) yields:

$$\Omega(s) = \frac{b}{s+a} \frac{M}{s} \qquad (16)$$

The inverse Laplace of (16) yields response in time-domain:

$$\omega(t) = \frac{Mb}{a}(1 - e^{-at}) \quad (17)$$

If we define:

$$a \triangleq \frac{1}{\tau} \quad (18)$$

and from NXT motor plot that steady-state, have

$$\lim_{t \rightarrow \infty} \omega(t) = \omega_{ss} \quad (19)$$

Then with (18) and (19), (17) yields:

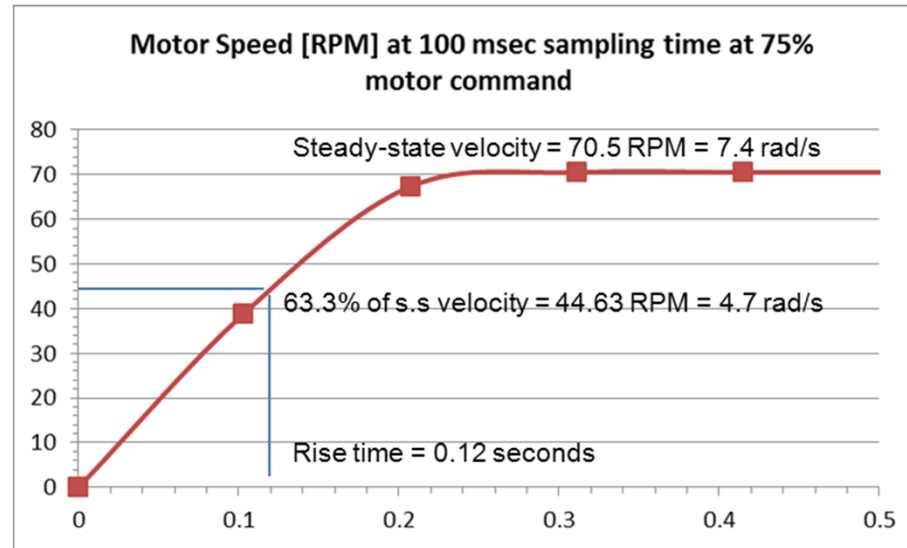
$$\omega_{ss} = Mb\tau \quad \text{or} \quad b = \frac{\omega_{ss}}{M\tau} \quad (17)$$

Hence, subbing values of  $a$  and  $b$  in (1), yields:

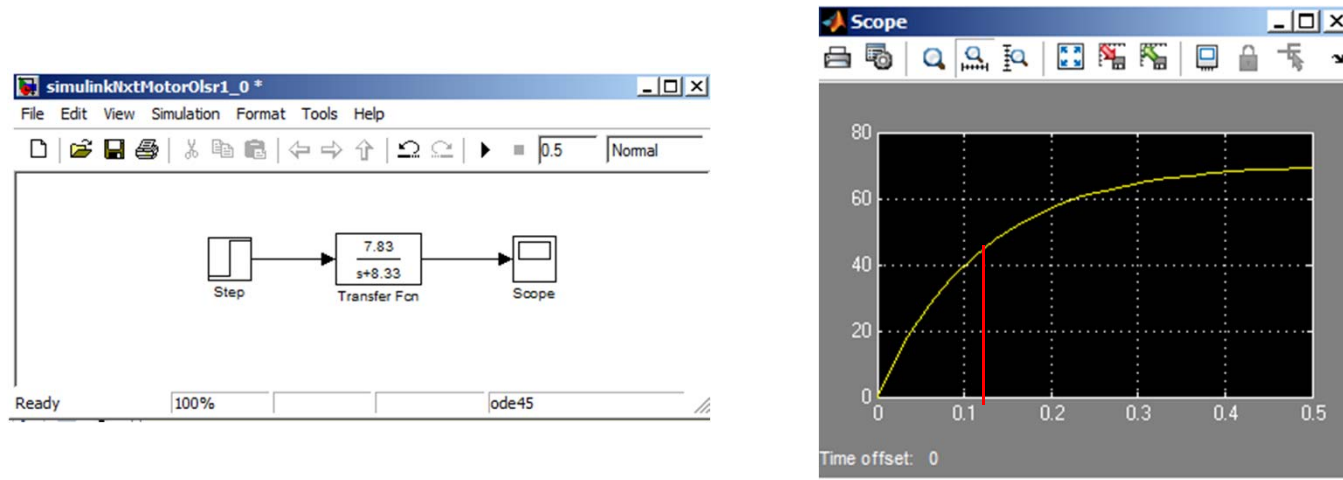
$$G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} = \frac{\omega_{ss}}{M\tau} \frac{1}{s + \frac{1}{\tau}} = \frac{70.5}{75(0.12)} \frac{1}{s + \frac{1}{0.12}} = \frac{7.83}{s + 8.33} \quad (18)$$

Same as (14); Laplace gives same  
Solution as ODE method





Simulink simulation of OLTF yields a plot similar to experimentally acquired one:



In Lab, will compare experimental and (Simulink) simulated OLTF step response