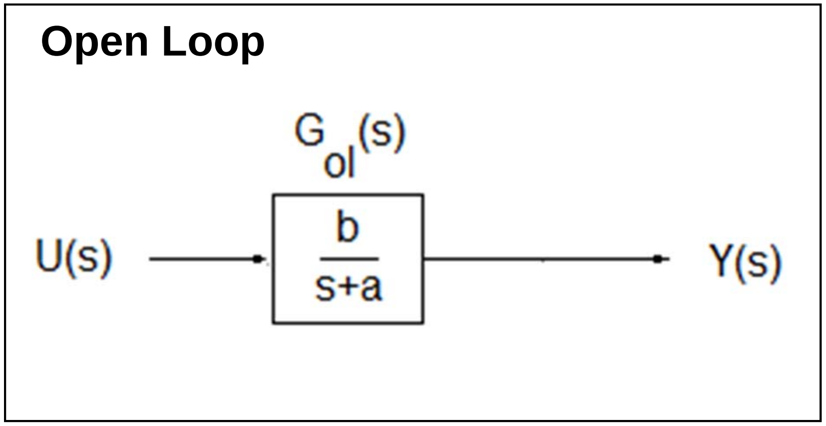
PID Control

Open Loop (no feedback)



$$Y(s) = G_{ol}(s)\, U(s)$$

$$Y(s) = \frac{b}{s+a} U(s)$$

Recall step input $U(s) \doteq \frac{1}{s}$

$$Y(s) = \frac{b}{s+a}\frac{1}{s}$$

Need to calculate [Inverse Laplace](https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119073444.app2)

$$\mathcal{L}^{-1}\left\{\frac{b}{s(s+a)}\right\}$$

Hence: $y(t) = \frac{b}{a}[1 - e^{-at}]$

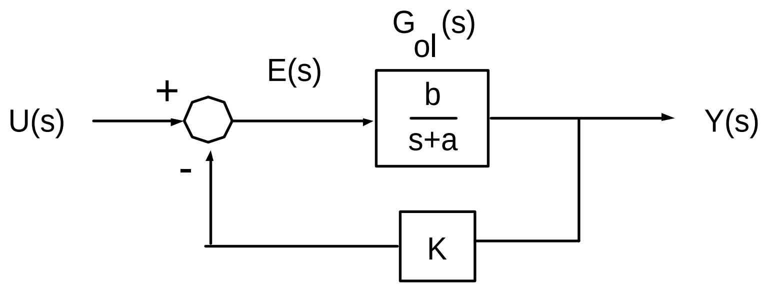
Steady-state: $$y_{ss} = \frac{b}{a} \qquad (1)$$

In other words, y_{ss} can never be $u(t) = 1$ (at least when a $\neq b$)

https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119073444.app2

Feedback (even Unity) is better than open-loop

Unity Feedback $(\boldsymbol{K=1})$



$$Y(s) = G_{ol}(s)E(s) = G_{ol}(s)\{U(s) - KY(s)\}$$

$$Y(s) = G_{ol}(s)U(s) - KG_{ol}(s)Y(s)$$

$$Y(s) + KG_{ol}(s)Y(s) = G_{ol}(s)U(s)$$

$$Y(s)\{1 + KG_{ol}(s)\} = G_{ol}(s)U(s)$$

$$Y(s) = \frac{G_{ol}(s)}{1 + KG_{ol}(s)}U(s)$$

$$Y(s) = \frac{\frac{b}{s+a}}{1 + K\frac{b}{s+a}}U(s) = \frac{b}{s+a+Kb}U(s)$$

Recall step input $U(s) \doteq \frac{1}{s}$

Need to calculate [Inverse Laplace](https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119073444.app2) $\mathcal{L}^{-1}\left\{\frac{b}{s+a+Kb}\cdot\frac{1}{s}\right\}$

Hence: $y(t) = \frac{b}{a+Kb}\left[1 - e^{-(a+Kb)t}\right]$

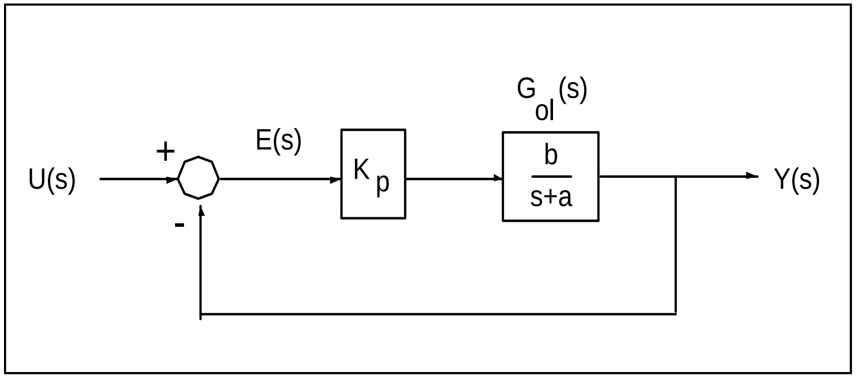
Steady-state: $$y_{ss} = \frac{b}{a+Kb} \quad (2)$$

As feedback gain increases, output attenuates

Again, y_{ss} can never be $u(t) = 1$ (at least when a $\neq b$) but better than (1)

https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119073444.app2

Proportional Even Better Than Unity Feedback



$$Y(s) = G_{ol}(s)K_pE(s) = G_{ol}(s)K_p\{U(s) - Y(s)\}$$

$$Y\{1 + G_{ol}K_p\} = G_{ol}K_pU$$

$$Y(s) = \frac{G_{ol}K_p}{1 + G_{ol}K_p} = \frac{\frac{b}{s+a}K_p}{1 + \frac{b}{s+a}K_p} = \frac{K_pb}{s + a + K_pb}$$

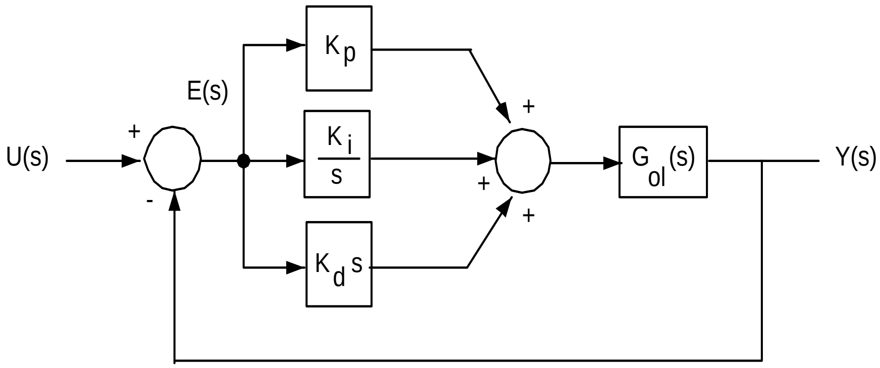
Inverse Laplace $y(t) = \frac{K_pb}{a + K_pb}\left[1 - e^{-(a+K_pb)t}\right]$ Steady-state: $y_{ss} = \frac{K_pb}{a + K_pb}$ (3)

As gain increases, proportional output reaches 1 (as desired)… but again, still never can be exactly 1

Comparing steady-state response to unit step input shows responses:

	Open Loop (1)	Feedback (2)	Proportional (3)
y_{ss} for step input	$\frac{b}{a}$	$\frac{b}{a + Kb}$	$\frac{K_pb}{a + K_pb}$

Proportional-Integral-Derivative Control



$$Y(s) = G_{ol}(K_p + \frac{K_i}{s} + K_d s)(U - Y)$$

- I improves steady-state accuracy at the expense of stability
- D improves stability at the expense of steady-state accuracy
- PD improves stability without degrading accuracy much
- PI improves steady-state accuracy without degrading stability much

$$G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{G_{ol}(K_p s + K_i + K_d s^2)}{s + G_{ol}(K_p s + K_i + K_d s^2)}$$

General PID CLTF

Case Study: Proportional only control

$$Y(s) = \frac{G_{ol}(K_p s)U(s)}{s + G_{ol}(K_p s)}$$

Step input $U(s) = \frac{1}{s}$

$$Y(s) = \frac{G_{ol}K_p}{s(1 + G_{ol}K_p)}$$

Final Value Theorem

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

$$y_{ss} = \lim_{s \to 0} sY(s) = \frac{G_{ol}K_p}{1 + G_{ol}K_p}$$

Response depends on OLTF poles and zeros

System Type: Keys to PID Design

$$y_{ss} = \lim_{s\to 0} sY(s) = \frac{G_{ol}K_p}{1+G_{ol}K_p}$$

Proportional only control

$$G_{ol}(s) = \frac{(\tau_a s+1)\cdots(\tau_m s+1)}{s^i(\tau_1 s+1)\cdots(\tau_n s+1)}$$

General form any TF

$i=0$	Type 0 System
$i=1$	Type 1 System
$i=2$	Type 2 System

(1)

Case 1: Type 0 System (step response)

$$y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{\frac{(\tau_a s+1)\cdots(\tau_m s+1)K_p}{(\tau_1 s+1)\cdots(\tau_n s+1)}}{\frac{(\tau_1 s+1)\cdots(\tau_n s+1)+(\tau_a s+1)\cdots(\tau_m s+1)K_p}{(\tau_1 s+1)\cdots(\tau_n s+1)}}$$

$$y_{ss} = \frac{K_p}{1+K_p}$$

NB: There is a discrete error, but if $K_p >> 1$ then $y_{ss} \approx 1$

Case 2: Type 1 System (step response)

$$y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{\frac{(\tau_a s+1)\cdots(\tau_m s+1)K_p}{s(\tau_1 s+1)\cdots(\tau_n s+1)}}{\frac{s(\tau_1 s+1)\cdots(\tau_n s+1)+(\tau_a s+1)\cdots(\tau_m s+1)K_p}{s(\tau_1 s+1)\cdots(\tau_n s+1)}}$$

$$y_{ss} = \frac{K_p}{0+K_p}$$

No error! Type 1 and 2 systems have **no error** with step response

Goal: Increase System Type with Integrator

$$G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{G_{ol}(K_p s + K_i + K_d s^2)}{s + G_{ol}(K_p s + K_i + K_d s^2)}$$

General PID CLTF

Case 1: Pure Integrator $K_i = 1$ $\quad Y(s) = \frac{G_{ol}U(s)}{s + G_{ol}}$ $\quad$ Step input $U(s) = \frac{1}{s}$

Substituting OLTF general form from (1)

$$Y(s) = \frac{\frac{(\tau_a s+1)\cdots(\tau_m s+1)}{s^i(\tau_1 s+1)\cdots(\tau_n s+1)} \cdot \frac{1}{s}}{s + \frac{(\tau_a s+1)\cdots(\tau_m s+1)}{s^i(\tau_1 s+1)\cdots(\tau_n s+1)}}$$

$$Y(s) = \frac{(\tau_a s+1)\cdots(\tau_m s+1) \cdot \frac{1}{s}}{[s \cdot s^i(\tau_1 s+1)\cdots(\tau_n s+1)] + (\tau_a s+1)\cdots(\tau_m s+1)}$$

$$y_{ss} = \lim_{s \to 0} sY(s) = \frac{(\tau_a s+1)\cdots(\tau_m s+1)}{[s \cdot s^i(\tau_1 s+1)\cdots(\tau_n s+1)] + (\tau_a s+1)\cdots(\tau_m s+1)}$$

Thus, adding an integrator yields $y_{ss} = 1$ **irregardless** of system type

Goal: Decrease System Type with Derivative

$$G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{G_{ol}(K_p s + K_i + K_d s^2)}{s + G_{ol}(K_p s + K_i + K_d s^2)}$$

General PID CLTF

Case 2: Pure Derivative $K_d = 1$

$$Y(s) = \frac{G_{ol} s^2 U(s)}{s + G_{ol} s^2}$$ Step input $U(s) = \frac{1}{s}$ then $$Y(s) = \frac{G_{ol}}{1 + G_{ol} s}$$

Substituting OLTF general form from (1)

$$Y(s) = \frac{\dfrac{(\tau_a s+1)\cdots(\tau_m s+1)}{s^i(\tau_1 s+1)\cdots(\tau_n s+1)}}{\dfrac{s^i(\tau_1 s+1)\cdots(\tau_n s+1) + (\tau_a s+1)\cdots(\tau_m s+1)s}{s^i(\tau_1 s+1)\cdots(\tau_n s+1)}}$$

$$y_{ss} = \lim_{s\to 0} sY(s) = \frac{s(\tau_a s+1)\cdots(\tau_m s+1)}{s^i(\tau_1 s+1)\cdots(\tau_n s+1) + (\tau_a s+1)\cdots(\tau_m s+1)s}$$

$$y_{ss} = \lim_{s\to 0} sY(s) = \frac{(\tau_a s+1)\cdots(\tau_m s+1)}{s^{i-1}(\tau_1 s+1)\cdots(\tau_n s+1) + (\tau_a s+1)\cdots(\tau_m s+1)}$$

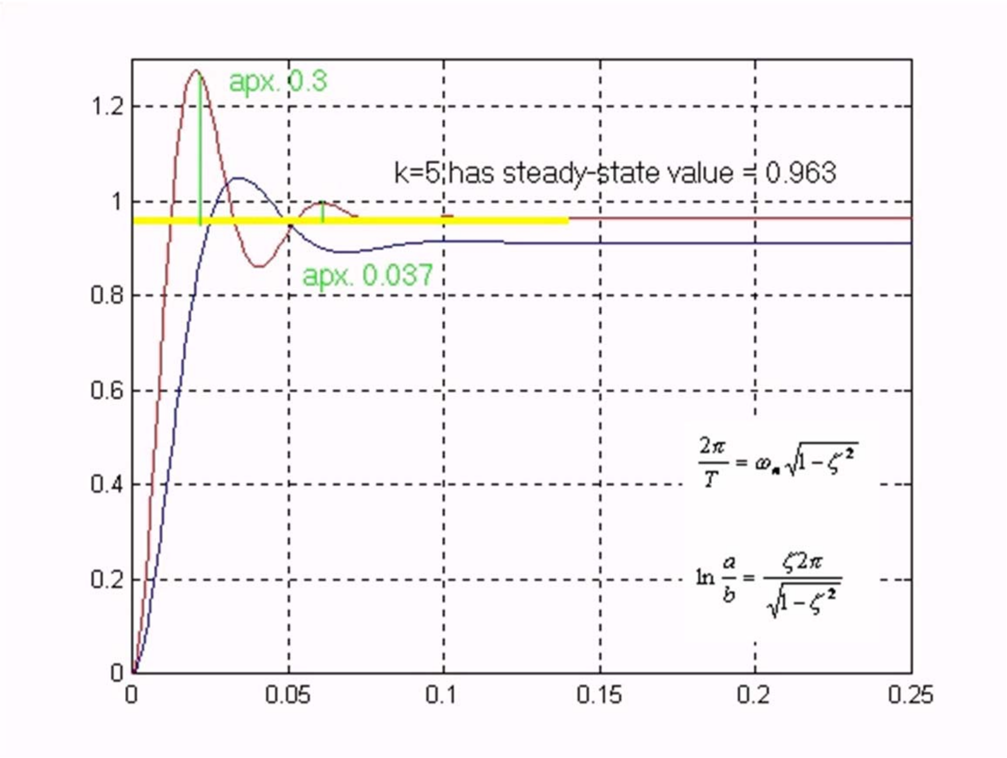
Derivative will yield $y_{ss} = 1$ if $i > 1$

Case Studies: Matlab Simulations

Recall motor given by: $$G_{ol} = \frac{5200}{(s+10)(s+100)}$$ Type 0 system

For $K_p = 2$ $$G_{cl}(s) = \frac{10400}{s^2 + 110s + 11400}$$

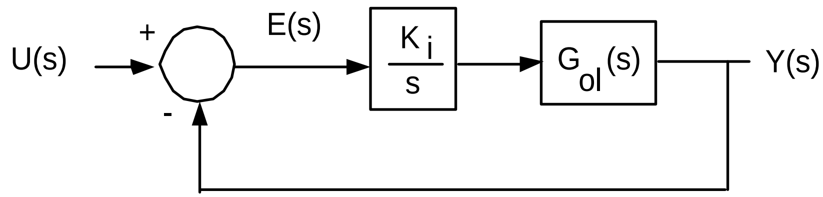
For $K_p = 5$ $$G_{cl}(s) = \frac{26000}{s^2 + 110s + 27000}$$



Matlab confirmed that NON-ZERO steady-state error (see mtr_p.m)

Case Studies: Matlab Simulations – Pure Integrator

Problem: Given



$$G_{ol} = \frac{5200}{(s+10)(s+100)}$$

$$Y(s) = G_{ol}(s)\frac{K_i}{s}(U - Y)$$

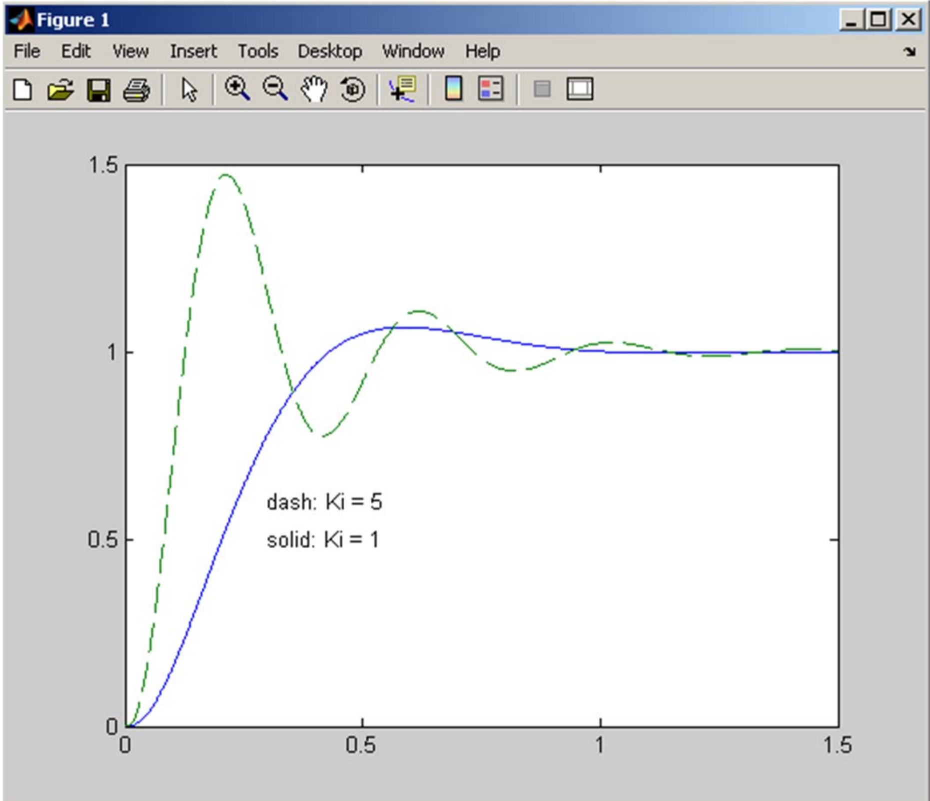
1-1: Show that the CLTF is

$$G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{5200K_i}{s(s+10)(s+100)+5200K_i}$$

1-2: Show that for a unit step response that $y_{ss} = 1$

1-3: In Matlab, simulate and display the unit step response with $K_i = 1$ and $K_i = 5$

Pure Integrator Control

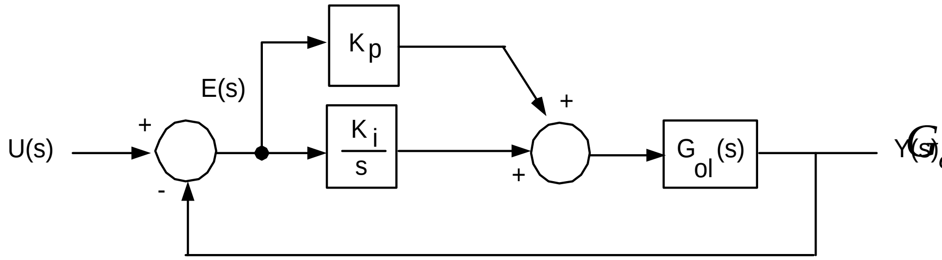


Code snippet

```
numki1 = [0   0   0 5200];     % numerator for Ki = 1
denki1 = [1 110 1000 5200]; % denominator

numki5 = [0   0   0 26000];     % numerator for Ki = 1
denki5 = [1 110 1000 26000]; % denominator

tFinal = input('Time range e.g. enter 0.5: ');
freq = input('Sampling frequency in Hz e.g. enter 1000: ');
tStep = 1/freq;
```

Case Studies: Matlab Simulations – PI control

Problem: Given



$$G_{ol} = \frac{5200}{(s+10)(s+100)}$$

2-1: Show that the CLTF is

$$G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{5200K_p s + 5200K_i}{s^3 + 110s^2 + (1000 + 5200K_p)s + 5200K_i}$$

2-2: In Matlab, simulate and display the unit step response with

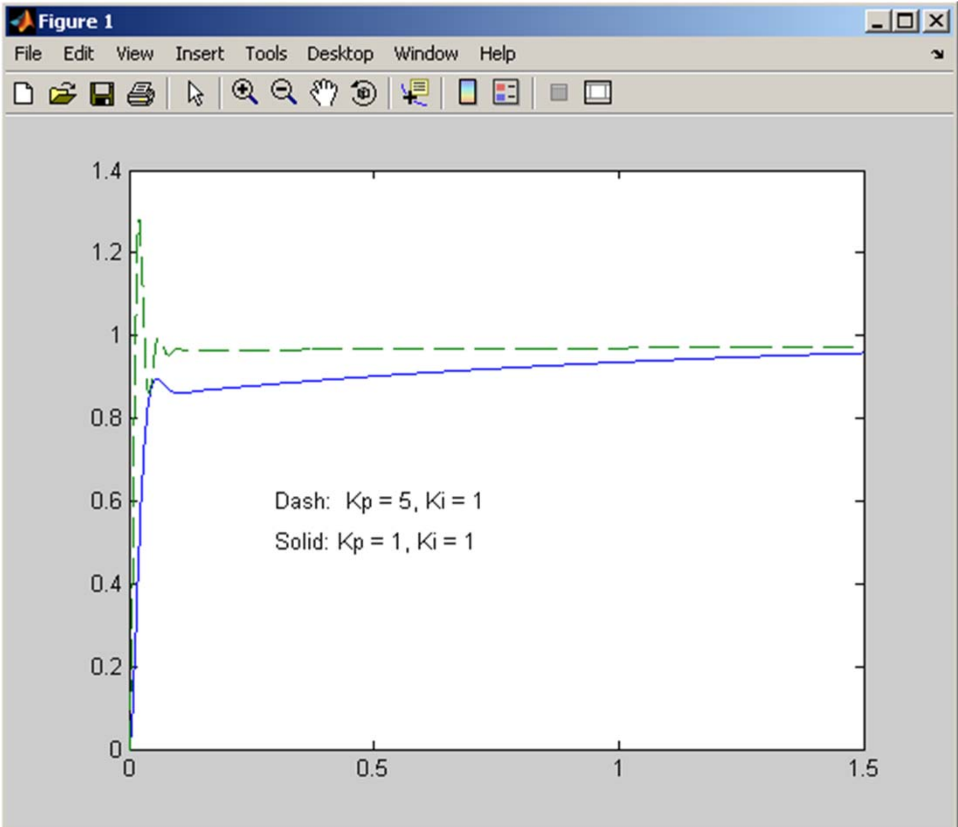
A. $K_i = 1 \quad K_p = 1$

B. $K_i = 1 \quad K_p = 5$

C. $K_i = 5 \quad K_p = 1$

D. $K_i = 5 \quad K_p = 5$

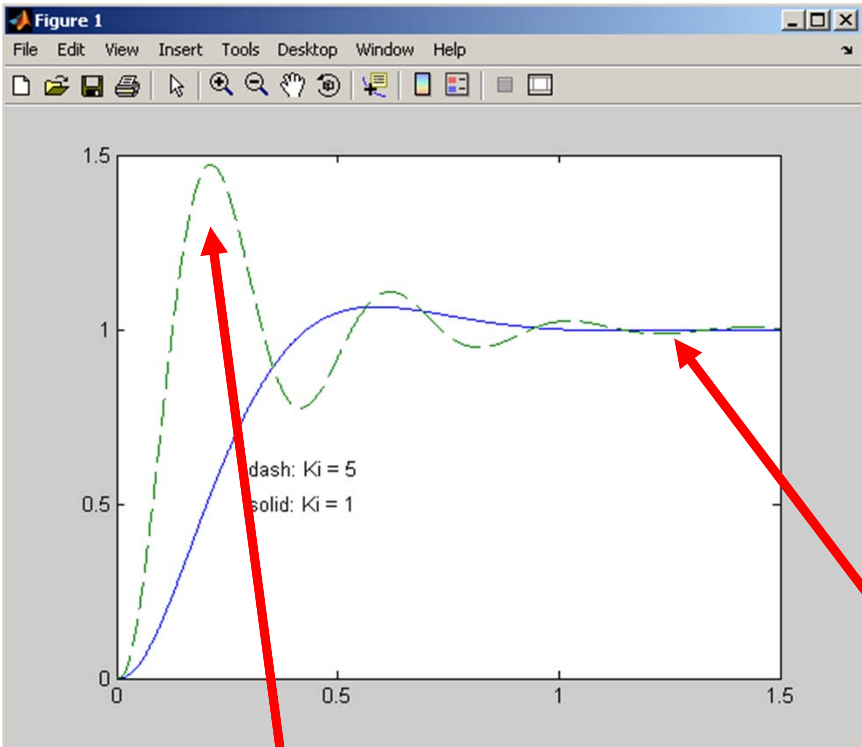
PI Control



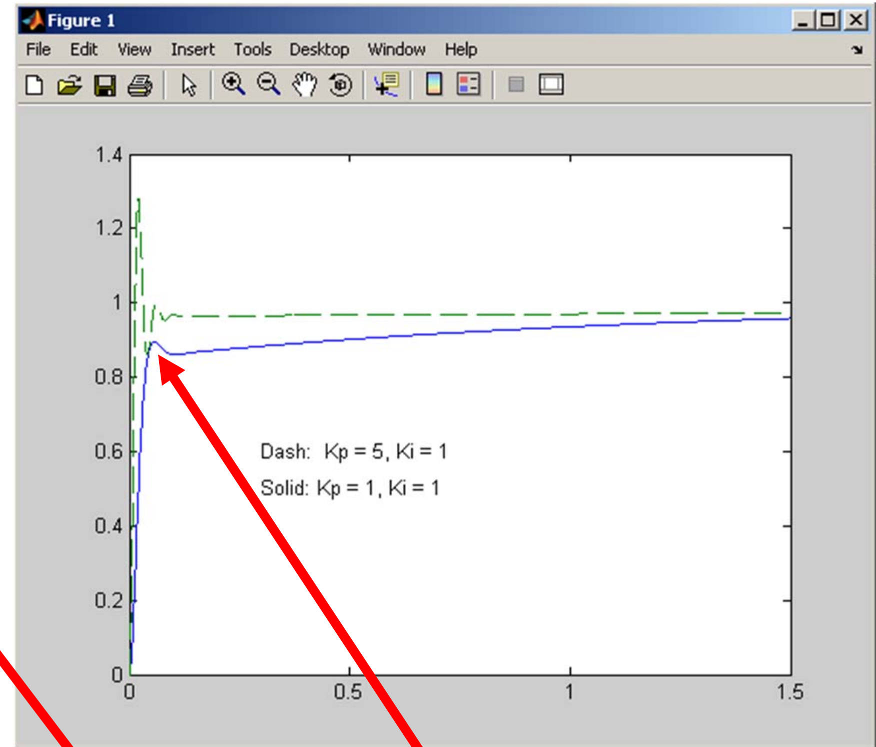
Code snippet

```
numkp5ki1 = [0 0   26000 5200]; % numerator for Kp = 5, Ki = 1
denkp5ki1 = [1 110 27000 5200]; % denominator

numkp1ki1 = [0 0   5200 5200]; % numerator for Kp = 1 , Ki = 1
denkp1ki1 = [1 110 6200 5200]; % denominator

tFinal = input('Time range e.g. enter 0.5: ');
freq = input('Sampling frequency in Hz e.g. enter 1000: ');
tStep = 1/freq;
```

Conclusions



Pure Integrator Control



PI Control

- I improves steady-state accuracy at the expense of stability
- PI improves steady-state accuracy without degrading stability much

Just like was stated in the beginning (see Slide3)