ME729 Advanced Robotics - Robot Dynamics

3/19/2018

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Lagrangian Mechanics

- Newton-Euler mechanics is a "force balance" approach to dynamics.
- Lagrangian mechanics is an "energy-based" approach to dynamics.
- The Lagrangian L is defined as the difference between the kinetic energy K and the potential energy P of the system.

$$L = K - P$$

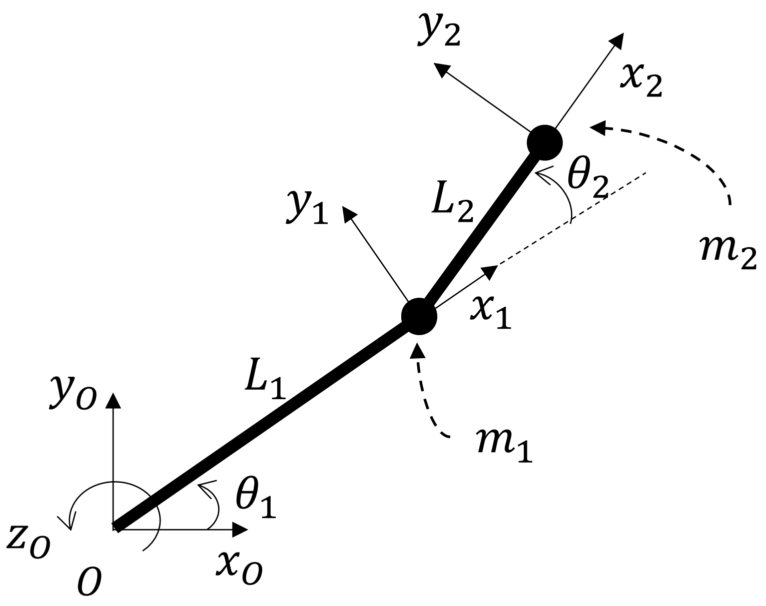
- The dynamics equations, in terms of the coordinates used to express the kinetic and potential energy, are obtained as

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

  where q_i are the coordinates in which the kinetic and potential energy are expressed, $\dot{q}_i$ is the corresponding velocity, and F_i the corresponding force or torque.

Lagrangian Mechanics

- For example, dynamics of the 2-link manipulator
  - No gravity effect.



- For link 1, the kinetic and potential energies are

$$K_1 = 1/2\, m_1 L_1^2 \dot{\theta}_1^2$$
$$P_1 = 0$$

- For link 2, we have

$x_2 = L_1 c_1 + L_2 c_{12}, \quad y_2 = L_1 s_1 + L_2 s_{12}$

$\dot{x}_2 = -L_1 \dot{\theta}_1 s_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2) s_{12}, \quad \dot{y}_2 = L_1 \dot{\theta}_1 c_1 + L_2 (\dot{\theta}_1 + \dot{\theta}_2) c_{12}$

so that the velocity squared is

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$
$$= L_1^2 \dot{\theta}_1^2 + L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) c_2$$

Therefore, the kinetic and potential energies for link 2 is

$K_2 = 1/2\, m_2 v_2^2$

$= 1/2\, m_2 L_1^2 \dot{\theta}_1^2 + 1/2\, m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) c_2$

$P_2 = 0$

Lagrangian Mechanics

- The Lagrangian

$$L = K_1 + K_2 - P_1 - P_2$$
$$= 1/2\,(m_1 + m_2)L_1^2\dot{\theta}_1^2 + 1/2\,m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1L_2\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2\right)c_2$$

- Lagrangian's equation

Link 1

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2\dot{\theta}_1 + m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2L_1L_2\left(2\dot{\theta}_1 + \dot{\theta}_2\right)c_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2L_1L_2\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)c_2 - m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2$$
$$\frac{\partial L}{\partial \theta_1} = 0$$

Link 2

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2L_1L_2\dot{\theta}_1c_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2L_1L_2\ddot{\theta}_1c_2 - m_2L_1L_2\dot{\theta}_1\dot{\theta}_2s_2$$
$$\frac{\partial L}{\partial \theta_2} = -m_2L_1L_2\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2\right)s_2$$

Lagrangian Mechanics

- Lagrangian's equation – continued

  $\therefore \tau_1 = \{(m_1 + m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2c_2\}\ddot{\theta}_1 + \{m_2L_2^2 + m_2L_1L_2c_2\}\ddot{\theta}_2 - m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2$

  $\tau_2 = \{m_2L_2^2 + m_2L_1L_2c_2\}\ddot{\theta}_1 + m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\dot{\theta}_1^2 s_2$

- A manipulator's equations of motion can be written in the form

$$\tau = M(\theta)\ddot{\theta} + V\left(\theta, \dot{\theta}\right) + G(\theta),$$

  where $M(\theta)$ is the n x n mass matrix, $V\left(\theta, \dot{\theta}\right)$ is an n x 1 vector of centrifugal and Coriolis terms, and $G(\theta)$ is and n x 1 vector of gravity terms.

- $\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1 + m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2c_2 & m_2L_2^2 + m_2L_1L_2c_2\\ m_2L_2^2 + m_2L_1L_2c_2 & m_2L_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}-m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2\\ m_2L_1L_2\dot{\theta}_1^2 s_2\end{bmatrix}$