ME729 Advanced Robotics - Homework #4 Solution

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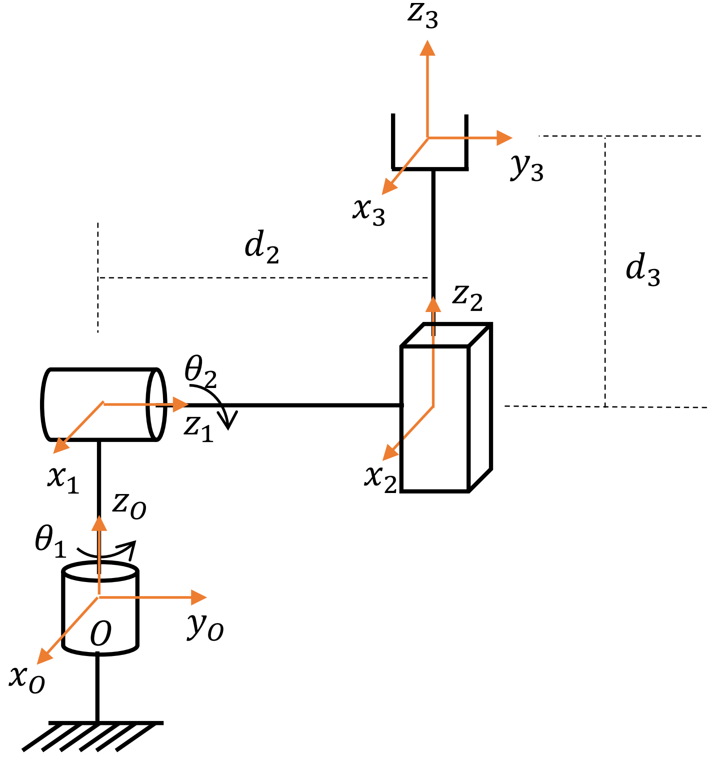
Email me *a pdf file* by next Monday 6 p.m.

1. A 3-dof manipulator has the following DH parameters and forward kinematics. [4]



i	θ_i	α_i	a_i	d_i
1	θ_1(t)	-90°	0	0
2	θ_2(t)	90°	0	d_2
3	0	0°	0	d_3(t)

$$
{}^{0}T_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -d_2s_1 + d_3c_1s_2 \\ c_2s_1 & c_1 & s_1s_2 & d_2c_1 + d_3s_1s_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$^*\cos(\theta_i) = c_i \, , \sin(\theta_i) = s_i$

1) Determine the Jacobian matrix $J(\boldsymbol{q})$ that relates the linear velocity of the end-effector $[\dot{x} \quad \dot{y} \quad \dot{z}]^T$ with the derivatives of the joint variable $\dot{\boldsymbol{q}} = [\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{d}_3]^T$ in the form of: [2]

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = J(\boldsymbol{q}) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \end{bmatrix}
$$

$$
J(\boldsymbol{q}) = \begin{bmatrix} -d_2c_1 - d_3s_1s_2 & d_3c_1c_2 & c_1s_2 \\ -d_2s_1 + d_3c_1s_2 & d_3s_1c_2 & s_1s_2 \\ 0 & -d_3s_2 & c_2 \end{bmatrix}
$$

2) Determine the singularities of the robot, if any. (you should calculate the inverse of a 3x3 matrix.) [2]

* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

$$\det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\therefore \det(A) = aei - afh - bdi + bfg + cdh - ceg$$

* $\det(J) = (-d_2c_1 - d_3s_1s_2)d_3s_1c_2^2 - (-d_2c_1 - d_3s_1s_2)(-d_3s_1s_2^2)$
$\quad -d_3c_1c_2^2(-d_2s_1 + d_3c_1s_2) + (-d_3c_1s_2^2)(-d_2s_1 + d_3c_1s_2)$

$$= -d_2d_3s_1c_1c_2^2 - d_3^2s_1^2s_2c_2^2 - d_2d_3s_1c_1s_2^2 - d_3^2s_1^2s_2^3 + d_2d_3s_1c_1c_2^2 - d_3^2c_1^2s_2c_2^2 + d_2d_3s_1c_1s_2^2 - d_3^2c_1^2s_2^3$$

$$= -d_3^2s_2c_2^2(s_1^2 + c_1^2) - d_3^2s_2^3(s_1^2 + c_1^2)$$

$$= -d_3^2s_2(s_2^2 + c_2^2)$$

$$= -d_3^2\sin(\theta_2)$$

When $d_3^2 = 0$ or $\sin(\theta_2) = 0$, then $-d_3^2\sin(\theta_2) = 0$.

$\therefore d_3 = 0$ or $\theta_2 = 0$ or $\theta_2 = \pi$

2. There are two cubics which are connected in a two-segment spline with continuous velocity and acceleration at the intermediate via point. The initial angle is θ_0, the via point is θ_v, and the goal point is θ_g. The initial and goal velocities are zeros.

The first cubic is

$$\theta_1(t) = a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10},$$

and the second cubic is

$$\theta_2(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}.$$

Each cubic will be evaluated over an interval starting at $t = 0$ and ending at $t = t_f$. [3.0]

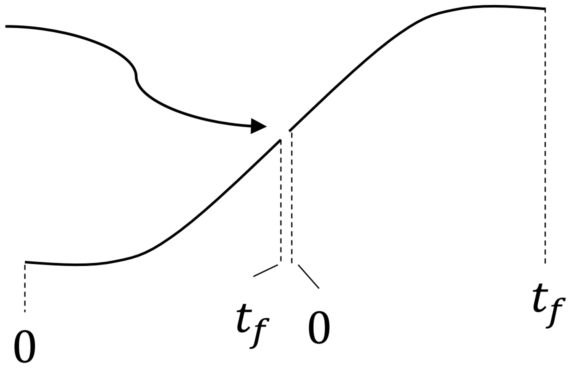
1) Find constraints. [1.5]

(1) Position constraints: $\theta_1(0) = \theta_0, \theta_1(t_f) = \theta_v, \theta_2(0) = \theta_v, \theta_2(t_f) = \theta_g$

(2) Velocity constraints: $\dot{\theta}_1(0) = 0, \dot{\theta}_1(t_f) = \dot{\theta}_2(0), \dot{\theta}_2(t_f) = 0$

(3) Acceleration constraints: $\ddot{\theta}_1(t_f) = \ddot{\theta}_2(0)$

$$\dot{\theta}_1(t_f) = \dot{\theta}_2(0)$$
$$\ddot{\theta}_1(t_f) = \ddot{\theta}_2(0)$$



2) Solve for the coefficients of two cubics. [1.5]

(1) $a_{10} = \theta_0$
(2) $a_{13}t_f^3 + a_{12}t_f^2 + a_{11}t_f + a_{10} = \theta_v$
(3) $a_{20} = \theta_v$
(4) $a_{23}t_f^3 + a_{22}t_f^2 + a_{21}t_f + a_{20} = \theta_g$
(5) $a_{11} = 0$
(6) $3a_{13}t_f^2 + 2a_{12}t_f + a_{11} = a_{21}$
(7) $3a_{23}t_f^2 + 2a_{22}t_f + a_{21} = 0$
(8) $6a_{13}t_f + 2a_{12} = 2a_{22}$

$$\longrightarrow \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t_f^3 & t_f^2 & t_f & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t_f^3 & t_f^2 & t_f & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3t_f^2 & 2t_f & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 3t_f^2 & 2t_f & 1 & 0 \\ 6t_f & 2 & 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_{13} \\ a_{12} \\ a_{11} \\ a_{10} \\ a_{23} \\ a_{22} \\ a_{21} \\ a_{20} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \theta_0 \\ \theta_v \\ \theta_v \\ \theta_g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B}$$

$\therefore X = A^{-1}B$

$$\therefore \quad a_{13} = \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3} \qquad a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}$$

$$a_{12} = \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2} \qquad a_{22} = \frac{-6\theta_v + 3\theta_g + 3\theta_0}{2t_f^2}$$

$$a_{11} = 0 \qquad a_{21} = \frac{3\theta_g - 3\theta_0}{4t_f}$$

$$a_{10} = \theta_0 \qquad a_{20} = \theta_v$$

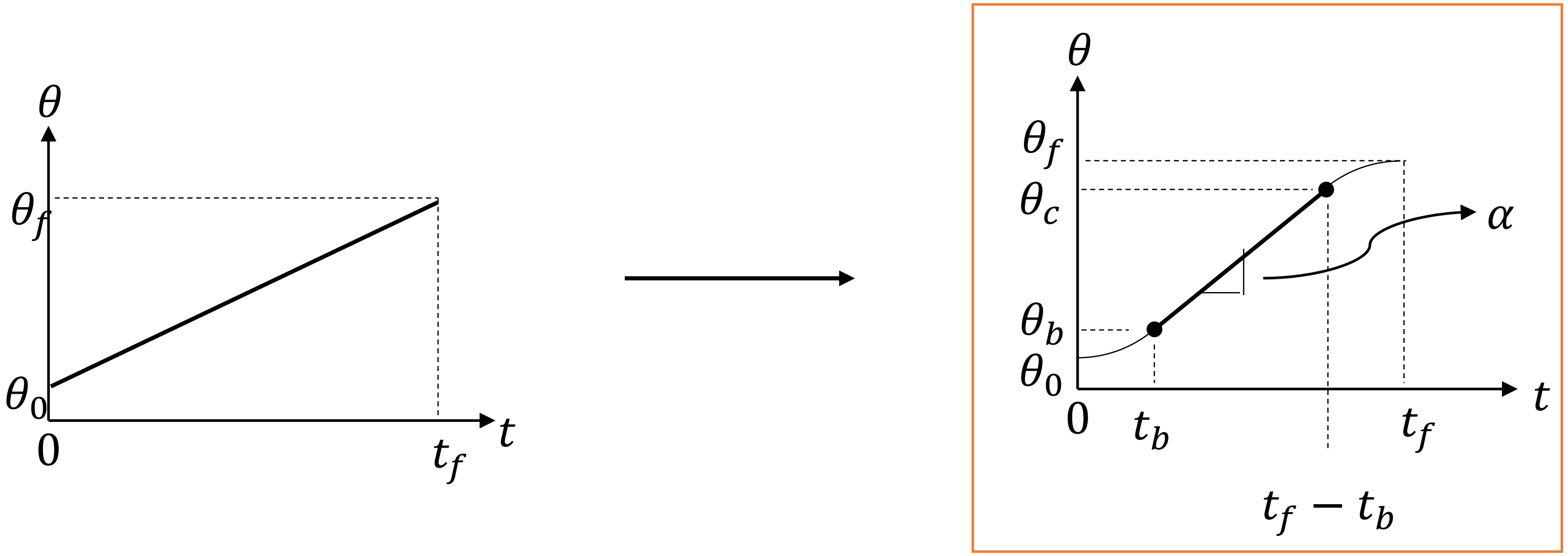
3. A linear path with parabolic blends is presented in figure. The initial and goal velocities are zero. In addition, the linear function and the two parabolic functions are splined together so that the entire path is continuous in position and velocity. The rate of change of the linear function is α. [3.0]

1) Find constraints. [1.0]

- The path can be divided into three segments.
- $0 \le t \le t_b$: $\theta_1(t) = a_{12}t^2 + a_{11}t + a_{10}$
- $t_b < t < t_f - t_b$: $\theta_2(t) = a_{21}t + a_{20}$
- $t_f - t_b \le t \le t_f$: $\theta_3(t) = a_{32}t^2 + a_{31}t + a_{30}$

(1) Position constraints: $\theta_1(0) = \theta_0,\ \ \theta_1(t_b) = \theta_b,\ \ \theta_3(t_f - t_b) = \theta_c,\ \ \theta_3(t_f) = \theta_f$

(2) Velocity constraints: $\dot{\theta}_1(0) = 0,\ \ \dot{\theta}_1(t_b) = \dot{\theta}_3(t_f - t_b),\ \ \dot{\theta}_3(t_f) = 0$



A linear path with parabolic blends

2) What is the trajectory equations? [1.0]

$$* \quad \theta_1(0) = a_{10} = \theta_0$$
$$\theta_1(t_b) = a_{12}t_b^2 + a_{11}t_b + a_{10} = \theta_b$$
$$\theta_3(t_f - t_b) = a_{32}(t_f - t_b)^2 + a_{31}(t_f - t_b) + a_{30} = \theta_c$$
$$\theta_3(t_f) = a_{32}t_f^2 + a_{31}t_f + a_{30} = \theta_f$$

$$* \quad \dot{\theta}_1(0) = a_{11} = 0$$
$$\dot{\theta}_1(t_b) = 2a_{12}t_b + a_{11} = a_{21}$$
$$\dot{\theta}_3(t_f - t_b) = 2a_{32}(t_f - t_b) + a_{31} = a_{21}$$
$$\dot{\theta}_3(t_f) = 0$$

$$* \quad \theta_b = a_{21}t_b + a_{20}$$
$$\theta_c = a_{21}(t_f - t_b) + a_{20}$$

$$\begin{cases} a_{12}t_b^2 - a_{21}t_b - a_{20} = -\theta_0 \\ -a_{21}(t_f - t_b) - a_{20} + a_{32}(t_f - t_b)^2 + a_{31}(t_f - t_b) + a_{30} = 0 \\ a_{32}t_f^2 + a_{31}t_f + a_{30} = \theta_f \\ 2a_{12}t_b - a_{21} = 0 \\ -a_{21} + 2a_{32}(t_f - t_b) + a_{31} = 0 \\ 2a_{32}t_f + a_{31} = 0 \end{cases}$$

$\longrightarrow$

$$a_{12} = \frac{-\theta_0 + \theta_f}{2t_b(t_f - 1)}$$
$$a_{11} = 0$$
$$a_{10} = \theta_0$$
$$a_{21} = \frac{-\theta_0 + \theta_f}{t_f - 1}$$
$$a_{20} = \frac{(2t_f - t_b)\theta_0 + (t_b - 2)\theta_f}{2(t_f - 1)}$$
$$a_{32} = \frac{\theta_0 - \theta_f}{2t_b(t_f - 1)}$$
$$a_{31} = \frac{-t_f(\theta_0 - \theta_f)}{t_b(t_f - 1)}$$
$$a_{30} = \frac{t_f^2(\theta_0 - \theta_f)}{2t_b(t_f - 1)} + \theta_f$$

2) What is the trajectory equations? [1.0] – continued

$$\therefore \theta(t) = \begin{cases} \dfrac{-\theta_0 + \theta_f}{2t_b(t_f - 1)} t^2 + \theta_0 \quad (0 \le t \le t_b) \\ \dfrac{-\theta_0 + \theta_f}{t_f - 1} t + \dfrac{(2t_f - t_b)\theta_0 + (t_b - 2)\theta_f}{2(t_f - 1)} \quad (t_b < t < t_f \; - t_b) \\ \dfrac{\theta_0 - \theta_f}{2t_b(t_f - 1)} t^2 + \dfrac{-t_f(\theta_0 - \theta_f)}{t_b(t_f - 1)} t + \dfrac{t_f^2\left(\theta_0 - \theta_f\right)}{2t_b(t_f - 1)} + \theta_f \quad (t_f - t_b \le t \le t_f) \end{cases}$$

3) Sketch graphs of velocity for the trajectory. [1.0]

$$\alpha = \frac{-\theta_0 + \theta_f}{t_f - 1}$$